# THE LAGRANGE SPECTRUM OF SOME SQUARE-TILED SURFACES

PASCAL HUBERT, SAMUEL LELIÈVRE, LUCA MARCHESE, CORINNA ULCIGRAI

Abstract. Lagrange spectra have been defined for closed submanifolds of the moduli space of translation surfaces which are invariant under the action of $\mathrm{SL}(2,\mathbb{R})$. We consider the closed orbit generated by a specific covering of degree 7 of the standard torus, which is an element of the stratum $\mathcal{H}(2)$. We give an explicit formula for the values in the spectrum, in terms of a cocycle over the classical continued fraction. Differently from the classical case of the modular surface, where the lowest part of the Lagrange spectrum is discrete, we find an isolated minimum, and a set with a rich structure right above it.

## 1. Introduction

The classical *Lagrange spectrum* $\mathcal{L}$ is a famous and well studied subset of the real line which admits both number theoretical and dynamical interpretations. It can be defined as the set of values of the function $L : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by

$$L(\alpha) := \limsup_{q,p\to\infty} \frac{1}{q|q\alpha - p|}$$

for $\alpha \in \mathbb{R}$. One has that $L(\alpha) = L < +\infty$ if and only if $\alpha$ *badly approximable*, or, more precisely if and only if for any $C > L$, we have $|\alpha - p/q| > (Cq^2)^{-1}$ for all $p$ and $q$ big enough and, moreover, $L$ is minimal with respect to this property. One can show that $\mathcal{L}$ can also be described as a *penetration spectrum* for the geodesic flow on the (unit tangent bundle of the) modular surface $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ in the following way. Let $(\gamma_t)_{t\in\mathbb{R}}$ be the projection to $\mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ of a hyperbolic geodesic on $\mathbb{H}$ which has $\alpha \in \mathbb{R}$ as forward endpoint. Then the value $L(\alpha)$ is a function of the *asymptotic depth of penetration* of the geodesic $\gamma_t$ into the cusp of the modular surface, that is given by

$$\limsup_{t\to+\infty} \mathrm{height}(\gamma_t),$$

where $\mathrm{height}(\cdot)$ denotes the *height* in the cusp of the modular surface (which, for an equivalence class $[z] \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$, is given by the supremum of the imaginary part $\Im z'$ over all $z'$ equivalent to $z$ under the $\mathrm{SL}(2,\mathbb{Z})$ action).

The structure of $\mathcal{L}$ has been studied for more than a century. Among the wide literature, we mention the very good introduction in [CuFl], where it is proved that $\mathcal{L}$ is a closed subset of the real line and that the values $L(\alpha)$ for $\alpha$ quadratic irrational form a dense subset. M. Hall in 1947 (see [Ha]) established that $\mathcal{L}(\mathcal{M})$ contains a positive half-line, also known as *Hall's ray*. The minimum of $\mathcal{L}$, known as *Hurwitz's constant*, is $\sqrt{5}$. The lower part of $\mathcal{L}$, that is the window $[\sqrt{5}, 3] \cap \mathcal{L}$, consists of a discrete sequence of quadratic irrational numbers which accumulate to 3, as proved by Markoff in 1879. The geometric picture corresponding to the discrete part of $\mathcal{L}$ was described in [Se]. A fine result on the Hausdorff dimension of $\mathcal{L}$ in the region between the discrete part and the Hall ray was proved more recently in [Mo].

A great variety of generalizations of Lagrange spectra have been defined and studied, mostly related to geodesic flows in negative curvature, in particular by Paulin-Parkkonen (see [ParPau] and references therein). We do no attempt to review the literature on the subject and we refer to [CuFl], [ParPau] and [HuMarUl] for a complete bibliography. The study of generalizations of the Lagrange spectrum has seen a very recent surge of interest, see for example the recent work by Ibarra and Moreira on Lagrange spectra for the geodesic flow on surfaces with variable negative curvature [IbMo] and in particular the various recent works on Lagrange spectra in the context of *translation surfaces*. Lagrange spectra of translation surfaces were first introduced by Hubert, Marchese and Ulcigrai in [HuMarUl], then studied in [ArMarUl] and [BoDe] and they are also the object of interest of the present paper.

*Translation surfaces* are surfaces obtained by glueing a finite set of polygons in the plane, by identifying pairs of isometric parallel sides by translations (see the next subsection definitions), so that one gets a surface which carries a flat metric with conical singularities. The *moduli space* of translation surfaces consists of translation surfaces up to the equivalence relation obtained by cutting and pasting by parallel translations. A natural *action of* $\mathrm{SL}(2,\mathbb{R})$ (the group of 2 by 2 matrices with real entries and determinant one) on the moduli space of translation surfaces is induced by the linear action of matrices on polygons in the plane (see the next subsection).

In [HuMarUl], Lagrange spectra are associated to any *locus* of translation surfaces in their moduli space which is closed and invariant under the $\mathrm{SL}(2,\mathbb{R})$ action. Examples of $\mathrm{SL}(2,\mathbb{R})$-invariant loci are *strata*, i.e. loci formed by all translation surfaces with a prescribed type of conical singularities and *Teichmüller curves*, i.e. (unit tangent bundles of) hyperbolic surfaces which are closed $\mathrm{SL}(2,\mathbb{R})$ orbits. The translation surfaces whose $\mathrm{SL}(2,\mathbb{R})$ orbit is a *Teichmüller curve* are known as *Veech* (or *lattice*) *surfaces* and are rich in affine symmetries. The simplest example of a Veech surface is the square torus (whose $\mathrm{SL}(2,\mathbb{R})$ orbit is the modular surface, for which the Lagrange spectrum defined in [HuMarUl] reduces to the classical case) and their covers, known as *square tiled surfaces* (see below for the definition). Lagrange spectra of closed invariant loci of translation surfaces share several of the same qualitative properties with $\mathcal{L}$. For example each of these generalized Lagrange spectra is closed and equal to the closure of the countable set of values corresponding to closed *Teichmüller geodesics* (see Theorem 1.5 in [HuMarUl]). Moreover, in [ArMarUl] it is proved that $\mathcal{L}(\mathcal{M})$ contains a positive half-line, also called *Hall's ray*, for all those $\mathcal{M}$ containing a *Veech surface* $S$ (see also Theorem 1.6 of [HuMarUl], where the same result was proved for loci which contain a square-tiled surface). Recently, M. Boshernitzan and V. Delecroix computed the minimum of the Lagrange spectrum of a connected component of a stratum of translation surfaces (whose value is given in the next subsection after Equation (1.2)) and showed that it is an isolated point (see [BoDe]).

The aim of this paper is to study an explicit simple case of Lagrange spectrum in the context of translation surfaces, for which we can provide a fine analysis which displays interesting phenomena which makes it different than the classical case. We consider *square-tiled surfaces*, that is translation surfaces $S$ tiled by copies of the square $[0,1]^2$, which as reminded earlier generate an orbit $\mathrm{SL}(2,\mathbb{R})\cdot S$ which is closed in the stratum, and for any such surface we consider the Lagrange spectrum $\mathcal{L}(S)$, where $\mathcal{L}(S)=\mathcal{L}(S')$ if $S$ and $S'$ have the same orbit closure (see Equations (1.1) and (1.2) and the discussion below for more details). A formula to compute elements in $\mathcal{L}(S)$ using continued fractions was obtained in [HuMarUl], but in general it applies only to elements in the spectrum which are *large enough* (see Theorem 5.12 in [HuMarUl] and also Theorem 2.9 below). With such formula, together with the explicit knowledge of the $\mathrm{SL}(2,\mathbb{R})$ orbits of square-tiled surfaces whit small number of squares, one can see that if $S$ is any square-tiled surface whose number of squares is at most 5, then $\mathcal{L}(S)$ is an affine copy of the classical Lagrange spectrum. In this paper we study of the Lagrange spectrum $\mathcal{L}(S)$ arising for a square tiled surface $S$ consisting of 7 squares in the so-called *Orbit B7* in the stratum $\mathcal{H}(2)$ (see § 3.1 for details). As explained in the next subsection (see also Remark 3.1), the $\mathrm{SL}(2,\mathbb{R})$ orbit closure of this surface is one of the simplest orbit closures whose study is both non-trivial and accessible by renormalization techniques. In particular the formula with continued fraction stated in Theorem 2.9 applies to *all* values in the Lagrange spectrum $\mathcal{L}(S)$ of this square-tiled surface, and our main result (Theorem 1.1 in the next section) describes the initial structure of $\mathcal{L}(S)$. We compute in particular the smallest value $\phi_1$ of $\mathcal{L}(S)$ and prove that it is isolated, while we show that the second smallest value $\phi_2$ is not isolated. In particular, this shows that $\mathcal{L}(S)$ provides an example of a Lagrange spectrum whose lower part is richer than in the classical case and for which the minimum $\phi_1$ is different than the minimum of the corresponding connected component computed by M. Boshernitzan and V. Delecroix in [BoDe]. Moreover, our analysis provide further information on the intricate structure of the bottom part of $\mathcal{L}(S)$, i.e. we can describe two countable families of accumulation points above $\phi_2$ which seems to indicate that any interval whose left endpoint is $\phi_2$ intersects $\mathcal{L}(S)$ in a Cantor set.

Our methods are a generalization of the methods used by Cusick and Flahive to analyze the discrete part of the classical Lagrange spectrum (see § 1 of [CuFl]). We crucially exploit the renormalization formula appearing in Theorem 2.9 and Corollary 3.2, which consists of two factors. One of them is simply given by the same expression appearing in the classical formula with continued fraction. The other factor is called *multiplicity* and represents a geometric information on lifts to $S$ of closed curves in $\mathbb{T}^2$. By analyzing the structure of the Orbit B7, which can be encoded in a graph, we show that to study the bottom of the spectrum it is sufficient to focus only on a smaller and simpler subgraph. Essentially, this allows to reduce the study of the Lagrange spectrum to the analysis of a limsup function over a subshift of finite type (see Theorem 1.2). We believe that this type of framework has the potential to be applied in much greater generality in the study of other Lagrange spectra and that our present analysis, in addition to its intrinsic interest, can also provide a paradigm for future works.

**Definitions and main results.** A *translation surface* is a genus $g$ closed surface $S$ with a flat metric and a finite set $\Sigma$ of conical singularities $p_1, \dots, p_r$, the angle at each $p_i$ being an integer multiple of $2\pi$. An equivalent definition of translation surface $S$ is the datum $(M, w)$, where $M$ is a compact Riemann surface and $w$ is an holomorphic 1-form on $M$ having a zero at each $p_i$. The relation $k_1+\dots+k_r = 2g-2$ holds, where $k_1, \dots, k_r$ are the orders of the zeroes of $w$. The 1-form $w$ induces a non-zero area form $(i/2)w \wedge \bar{w}$ on $S$ and we fix a normalization on translation surfaces requiring $\mathrm{Area}(S) = 1$. A stratum $\mathcal{H} = \mathcal{H}(k_1, \dots, k_r)$ is the set of translation surfaces $S$ whose corresponding holomorphic 1-form $w$ has $r$ zeros with orders $k_1, \dots, k_r$, where $k_1 + \dots + k_r = 2g - 2$. Any stratum admits an action of $\mathrm{SL}(2, \mathbb{R})$, indeed for a translation surface $S = (M, w)$ and an element $G \in \mathrm{SL}(2, \mathbb{R})$ a new translation surface $G \cdot S = (G_*M, G_*w)$ is defined, where the 1-form $G_*w$ is the composition of $w$ with $G$ and $G_*M$ is the complex atlas for which $G_*w$ is holomorphic. In the following we mostly consider the action of the geodesic flow $g_t$ and the group of rotations $r_\theta$, defined respectively for $t \in \mathbb{R}$ and $-\pi \leq \theta < \pi$ by

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \text{ and } r_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

For a general overview on translation surfaces we recommend the surveys [FoMat] and [Zo].

A *saddle connection* for the translation surface $S$ is a geodesic $\gamma$ for the flat metric connecting two conical singularities and not containing points of $\Sigma$ in its interior. Any saddle connection corresponds to some relative homology class, nevertheless not any class in $H_1(S, \Sigma, \mathbb{Z})$ contains a saddle connection $\gamma$, indeed in general relative homology classes are represented by piecewise smooth paths, which are composition of several saddle connections $\gamma_1, \dots, \gamma_l$. The set $\mathrm{Hol}(S)$ of *relative periods* of $X$ is the set of complex numbers $v := \int_\gamma w$, where $\gamma$ is a saddle connection for $S$ and $w$ is the holomorphic 1-form. In the rest of the paper, elements of $\mathrm{Hol}(S)$ will be called simply *periods*. The systole function $\mathrm{Sys} : \mathcal{H} \to \mathbb{R}_+$ is defined by

$$\mathrm{Sys}(S) := \min_{v \in \mathrm{Hol}(S)} |v|.$$

Strata are non compact. An exhaustion by compact subsets is the family $(\mathcal{K})_{c>0}$, where for any $c > 0$ we define $\mathcal{K}_c$ as the set of those $S$ such that $\mathrm{Sys}(S) \geq c$. Therefore the positive $g_t$-orbit of a translation surface $S$ stays in a compact if $\mathrm{Sys}(g_t \cdot S) > c$ for some constant $c > 0$ and any $t > 0$. It is known from Vorobets, and reproved in Proposition 1.1 of [HuMarUl], that

$$L(S) := \limsup_{|\mathrm{Im}(v)| \to \infty} \frac{1}{|\mathrm{Re}(v)| \cdot |\mathrm{Im}(v)|} = \limsup_{t \to +\infty} \frac{2}{\left(\mathrm{Sys}(g_t \cdot S)\right)^2}, \tag{1.1}$$

where the lim sup in the middle term is taken over all periods $v \in \mathrm{Hol}(S)$. Therefore the quantity $L(S)$ defined above gives a measure of the size of the asymptotical maximal excursion of the positive $g_t$-orbit of $S$. Let $\mathcal{M}$ be an orbit closure for the action of $\mathrm{SL}(2, \mathbb{R})$ in some stratum. The associated *Lagrange spectrum* is the set of values

$$\mathcal{L}(\mathcal{M}) := \{L(S) \; ; \; S \in \mathcal{M}\}. \tag{1.2}$$

According to Theorem 1.5 of [HuMarUl], any $\mathcal{L}(\mathcal{M})$ is a closed subset of the real line, equal to the closure of the set of values $L(S)$ for those $S \in \mathcal{M}$ whose geodesic $g_t \cdot S$ is periodic. Moreover in [ArMarUl] it is proved that $\mathcal{L}(\mathcal{M})$ contains a positive half-line, also called *Hall's ray*, for all those $\mathcal{M}$ containing a *Veech surface* $S$ (see also Theorem 1.6 of [HuMarUl]). Recently, M. Boshernitzan and V. Delecroix proved in [BoDe] that if $\mathcal{M}$ is a connected component of any stratum $\mathcal{H}(k_1,\dots,k_r)$, then the minimum of $\mathcal{L}(\mathcal{M})$ is $(k_1+\dots+k_r+r)\sqrt{5}$ and is isolated.

In this paper we consider *square-tiled surfaces* $S$, also called *origamis*, that is translation surfaces $S$ tiled by copies of the square $[0,1]^2$. Equivalently, $S$ is square-tiled if there exists a ramified covering $\rho: S \to \mathbb{R}^2/\mathbb{Z}^2$, unramified outside $0 \in \mathbb{R}^2/\mathbb{Z}^2$ and such that $\rho^*(dz)$ is the holomorphic 1-form of $S$. Square tiled surfaces are examples of Veech surfaces and any orbit $\mathrm{SL}(2,\mathbb{R})\cdot S$ of a square-tiled surface $S$ is closed in its stratum. For simplicity we denote its Lagrange spectrum by $\mathcal{L}(S)$ instead of $\mathcal{L}(\mathrm{SL}(2,\mathbb{R})\cdot S)$, where of course $\mathcal{L}(S)=\mathcal{L}(S')$ if $S$ and $S'$ have the same orbit closure. Observe that it is implicit in our definition that the vertical and the horizontal directions on $S$ are rational directions. Consider any other direction and let $\theta$ be the angle it forms with the vertical, then set $\alpha := \tan\theta$. At page 187 of [HuMarUl] it is explained that $\mathcal{L}(S)$ is parametrized by the function $L(S,\cdot):\mathbb{R}\to\mathcal{L}(S)$ defined by

$$L(S,\alpha) := L(r_{\arctan(\alpha)}\cdot S).$$

It is an easy exercise to prove that for $S=\mathbb{T}^2$ we have

$$L(\mathbb{T}^2,\alpha) = \limsup_{q,p\to\infty} \frac{1}{q\cdot|q\alpha-p|},$$

which is the function giving rise to the classical Lagrange spectrum $\mathcal{L}$, that is the set of values $L(\mathbb{T}^2,\alpha)$ for $\alpha$ badly approximable (see for example Lemma 5.10 in [HuMarUl]).

In the present paper we study the Lagrange spectrum of the following set of origamis with 7 squares. The action of $\mathrm{SL}(2,\mathbb{R})$ induces an action of $\mathrm{SL}(2,\mathbb{Z})$ on square-tiled surfaces, and the latter preserves the set of origamis with a given number $N$ of squares (see § 2.2). In the stratum $\mathcal{H}(2)$, the orbits of such action have been classified in [HuLe] and [Mc], proving that for any integer $N$ there are always at most two orbits and when $N$ is odd the orbits are exactly two, denoted by Orbit AN and Orbit BN respectively (see § 2.2 for more details). The Orbit B7 is described explicitly in § 3.1.

The interest in studying the Orbit B7 can be seen from Figure 2 (which shows the square-tiled surfaces which belong to it): the orbit contains origamis $X_j$ such that any horizontal saddle connection of $X_j$ winds *at least* twice around the horizontal closed geodesic of $\mathbb{T}^2$ under the covering map $\rho: X_j\to\mathbb{T}^2$. One can see that if $\mathcal{O}$ is an orbit where such property is not satisfied, for example when the number of squares is at most 5, then $\mathcal{L}(\mathcal{O})$ is an affine copy of the classical Lagrange spectrum $\mathcal{L}(\mathbb{T}^2)$. On the other hand, any $X_j$ in the Orbit B7 always has an horizontal saddle connection that winds around the horizontal closed geodesic of $\mathbb{T}^2$ *at most* twice. When the latter property holds, one can apply a nice renormalization formula to compute all the elements of the Lagrange spectrum (and not just those which are big enough, see Theorem 2.9 and Remark 3.1), whereas we do not know if the formula holds for arbitrary number of squares, for which the latter property does not generally hold. The two properties mentioned above make the Orbit B7 one of the simplest orbit closures whose study is both non-trivial and accessible by renormalization techniques. The other orbits with at most 7 squares and satisfying the same two properties are the Orbit A7, which has 54 elements, and the orbit of square-tiled surfaces with 6 squares in $\mathcal{H}(2)$, which has 36 elements. Since the Orbit B7 has 36 elements, we chose it as the easiest case to study.

Our main result, namely Theorem 1.1 below, describes the structure of the bottom of the Lagrange spectrum $\mathcal{L}(S)$ arising from any origami $S$ within the B7 Orbit. In order to state it, let us recall some continued fractions notation. Any irrational real number $\alpha$ admits an unique continued fraction expansion $\alpha = a_0 + [a_1,a_2,\dots]$, where $a_0$ is the integer part of $\alpha$ and where

$$[a_1,a_2,\dots] := \cfrac{1}{a_1+\cfrac{1}{a_2+\dots}}.$$

It is well known that the sequence of entries $(a_n)_{n\in\mathbb{N}}$ is eventually periodic if and only if $\alpha$ is a quadratic irrational. In this case we write

$$\alpha = a_0 + [b_1, \dots, b_m, \overline{a_1, \dots, a_n}],$$

where $b_1, \dots, b_m$ is the pre-periodic part of the sequence of entries and $a_1, \dots, a_n$ is the period of the periodic part. Consider the following three positive values $\phi_1 < \phi_2 < \phi_\infty$, where

$$\phi_1 := 7 + 14 \cdot [\overline{3,1}] = \frac{7\sqrt{21}}{3} = 10,696277 \pm 10^{-6}$$
$$\phi_2 := 14 \cdot [1, 4, \overline{1,3}] = 14 \cdot \frac{4\sqrt{21}+18}{5\sqrt{21}+21} = 11,582576 \pm 10^{-6}$$
$$\phi_\infty := 14 \cdot [1, 4, \overline{1,4,2,4}] = 14 \cdot \frac{2\sqrt{210}+24}{2\sqrt{210}+35} = 11,593101 \pm 10^{-6}.$$

Consider a bounded interval $G$ and denote its endpoints by $G^{(-)} := \inf G$ and $G^{(+)} := \sup G$. A gap in $\mathcal{L}(S)$ is an open interval $G$ as above such that $G \cap \mathcal{L}(S) = \emptyset$ and $G^{(-)} \in \mathcal{L}(S)$ and $G^{(+)} \in \mathcal{L}(S)$. In particular, if $G$ is a gap in $\mathcal{L}(S)$ then its endpoints are elements of the spectrum with $G^{(-)} < G^{(+)}$. Different gaps are always disjoint. Our main result is the theorem below.

**Theorem 1.1.** *Let $S$ be any square-tiled surface in the Orbit B7 in $\mathcal{H}(2)$ and consider its Lagrange spectrum $\mathcal{L}(S)$. The following holds.*

(1) *We have $\min \mathcal{L}(S) = \phi_1$ and the latter is an isolated point of $\mathcal{L}(S)$. More precisely $G_0 := (\phi_1, \phi_2)$ is the first gap of $\mathcal{L}(S)$.*

(2) *There exists a sequence of gaps $(G_k)_{k\geq 0}$ above $G_0$ ordered so that for any $k \geq 0$ we have*

$$\phi_1 < \dots < G_k^{(+)} < G_{k+1}^{(-)} < \dots < \phi_\infty.$$

*Moreover the gaps $G_k$ accumulate to $\phi_\infty$, that is*

$$\lim_{k\to+\infty} G_k^{(+)} = \phi_\infty.$$

(3) *For any $k \geq 1$ the window $\mathcal{L}(S) \cap [G_k^{(+)}, G_{k+1}^{(-)}]$ contains a sequence of gaps $(G_{k,n})_{n\geq 1}$ in* reversed order*, namely such that for any $n \geq 1$ we have*

$$G_k^{(+)} < \dots < G_{k,n+1}^{(+)} < G_{k,n}^{(-)} < \dots < G_{k+1}^{(-)}.$$

*Moreover the gaps $G_{k,n}$ accumulate to $G_k^{(+)}$, that is*

$$\lim_{n\to+\infty} G_{k,n}^{(-)} = G_k^{(+)}.$$

*In particular $\phi_2$ is the second smallest value of $\mathcal{L}(S)$ and it is not an isolated point of the spectrum.*

(4) *There exists a gap $G_\infty$ in $\mathcal{L}(S)$ such that $G_\infty^{(-)} = \phi_\infty$.*

The main intermediate technical result towards the proof of Theorem 1.1 is Theorem 1.2 below. Consider the two finite words $a := 1,4,2,4$ and $b := 1,3$. Let $\Xi$ be the subset of $\{a,b\}^{\mathbb{Z}}$ of those sequences $\xi = (\xi_k)_{k\in\mathbb{Z}}$ such that for any $N$ there exists $n > N$ with $\xi_n = a$. Let $\Xi_0 \subset \Xi$ be the subset of those $\xi \in \Xi$ satisfying the extra condition $\xi_0 = a$. Let $\tilde{\sigma} : \Xi \to \Xi$ be the shift and consider its first return map $\sigma : \Xi_0 \to \Xi_0$. If $\omega$ is a semi-infinite sequence in $\{a,b\}^{\mathbb{N}}$, let $[\omega] \in (0,1)$ be the real number $\alpha$ such that the entries $a_n$ of the continued fraction $\alpha = [a_1, a_2, \dots]$ are the digits of the sequence in $\{1,2,3,4\}$ obtained from $w$ by concatanating the blocks $a, b$ in the order given by $\omega$. For example, if $w = (a, b, b, b, a, \dots)$, then

$$[\omega] = [a, b, b, b, a, \dots] = [\underbrace{1,4,2,4}_{a}, \underbrace{1,3}_{b}, \underbrace{1,3}_{b}, \underbrace{1,3}_{b}, \underbrace{1,4,2,4}_{a}, \dots].$$

Define two functions $[\cdot]_+ : \Xi_0 \to \mathbb{R}$ and $[\cdot]_- : \Xi_0 \to \mathbb{R}$ by

$$[\xi]_+ := [1, 4, \xi_1, \xi_2, \dots],$$
$$[\xi]_- := [1, 4, \xi_{-1}, \xi_{-2}, \dots],$$

where, with the notation introduced above, $\xi_k \in \{a, b\}$ represent blocks of digits in $\{1, 2, 3, 4\}$. Define also a function $L^\sigma : \Xi_0 \to \mathbb{R}_+$ by

$$L^\sigma(\xi) := 7 \cdot \limsup_{n \to +\infty} \left([\sigma^n(\xi)]_- + [\sigma^n(\xi)]_+\right).$$

Let $\eta_1$ be the quadratic irrational

$$\eta_1 = 7 \cdot \frac{[1, 4, 2, \overline{1, 5}] + 5 + [1, 5, 1, \overline{1, 5}]}{4} = 11,655309 \pm 10^{-6}$$

and observe that $\eta_1 > \phi_\infty$.

**Theorem 1.2.** *Consider data $(S', \alpha)$, where $S'$ is a square-tiled surface in the Orbit B7 and $\alpha \in \mathbb{R}$. If $L^\sigma(S', \alpha) > \phi_1$ then we have also $L(S', \alpha) \geq \phi_2$. Moreover, for those data $(S', \alpha)$ with $\phi_2 \leq L(S', \alpha) < \eta_1$ there exists $\xi \in \Xi_0$ such that*

$$L(S', \alpha) = L^\sigma(\xi).$$

Thus, Theorem 1.2 shows that the study of the *bottom* of the Lagrange spectrum of origamis in the Orbit B7 can be reduced to the study of the values of the function $L^\sigma$ on a subshift of finite type. More precisely, let $\mathbb{K}$ be the set of values of the function $L^\sigma : \Xi_0 \to \mathbb{R}_+$. By Theorem 1.2, $\mathcal{L}(S) \cap [\phi_2, \eta_1) = \mathbb{K}$ for any origami $S$ in the Orbit B7. This is useful since we will then introduce a lexicographic order on half-infinite words in the letters $a, b$, which induces an order on the values of the function $L^\sigma : \Xi_0 \to \mathbb{R}_+$ (see § 5.1). In this paper we could not completely describe the set $\mathbb{K}$, nevertheless we observed a certain self-similarity in it and we formulate the Conjecture below.

**Conjecture 1.3.** *We conjecture that $\mathbb{K}$ is a Cantor set, that is a closed subset of the real line with empty interior and no isolated points. Such a set is the complement of countably many disjoint open intervals, and the families of gaps $(G_k)_{k \geq 2}$ and $(G_{k,n})_{k \geq 2, n \geq 1}$ in Theorem 1.1 seem to give respectively the first and the second generation of an iterative construction.*

Besides our Conjecture 1.3, one can ask if the set $\mathbb{K}$ presents dimension properties similar to those established by Moreira in [Mo] for the region of the classical Lagrange spectrum $\mathcal{L}$ between the discrete part and the Hall ray. In particular Moreira proves that the function $t \mapsto HD(\mathcal{L} \cap (0, t))$ is continuous and increasing, where $HD(E)$ denotes the Hausdorff dimension of a set $E \subset \mathbb{R}$. More recently, in [CeMatMo], it was shown that such dimension properties are common to a large class of dynamically defined Lagrange spectra.

**Structure of the rest of the paper.** In § 2, the main result is a formula to compute values $L(S, \alpha)$ in $\mathcal{L}(S)$ in terms of the continued fraction expansion of $\alpha$ and a geometric factor called multiplicity (see Theorem 2.9). This formula holds for any reduced square-tiled surface $S$, but, in general, it allows to compute only values $L(S, \alpha)$ for slopes $\alpha$ such that $L(\mathbb{T}^2, \alpha)$ is bigger that a given value, which depends only on the orbit of $S$ under $\mathrm{SL}(2, \mathbb{Z})$. The starting point for this result is given by Lemma 2.8, which expresses $L(S, \alpha)$ in terms of small values of the quantity $m^2(p/q, S) \cdot q \cdot |q\alpha - p|$ when $p/q$ varies among rational numbers. The factor $m(p/q, S)$ is called the *multiplicity* of the slope $p/q$ and is defined in § 2.3, where some of its elementary properties are also proved. Some general facts about $\mathrm{SL}(2, \mathbb{Z})$-orbits are recalled in § 2.2.

In § 3 we describe the Orbit B7 of $\mathcal{H}(2)$ is explicitly (see § 3.1) and show that for any origami $S$ in the Orbit B7 of $\mathcal{H}(2)$ the formula of Theorem 2.9 gives the value $L(S, \alpha)$ *for any* $\alpha$ (see Corollary 3.2). In § 3.3 we consider the special case when $\alpha$ is a quadratic irrational , or equivalently we consider closed geodesics in moduli space, and we establish a formula (see Equation (3.2)), which in this case gives $L(S, \alpha)$ as the maximum over a finite set of values. This is very useful for a numerical study of the Lagrange spectrum, which we do not attempt in this paper.

In § 4 we prove Theorem 1.2. The Orbit B7 can be represented as an oriented graph, whose vertices are origamis in the orbit and whose oriented edges represent the action of the two generators $T$ and $R$ of $\mathrm{SL}(2,\mathbb{Z})$. To paths on this graph, one can associate Lagrange values and conversely each datum $(S,\alpha)$ produces a path with value $L(S,\alpha)$. The results in these sections all show that values of the spectrum $\mathcal{L}$ of the B7 Orbit below certain thresholds are only realized by paths which are composition of a restricted class of operations. More precisely, we introduce an *intermediate graph* $\mathcal{I}$ (whose vertices are 8 special elements of the Orbit B7 and whose arrows are 30 special combinations of the operations $T$ and $R$) and its subgraph $\mathcal{S}$, called *small graph* (which has 3 vertices and 6 arrows). In Proposition 4.7 we prove that values $L\in\mathcal{L}$ such that $L<\eta_3$, where $\eta_3$ is a quadratic irrational such that $\eta_3>\eta_1>\phi_\infty$, only come from paths which only contains operations appearing in the intermediate graph. Then Proposition 4.3 allows to reduce to considering paths on the small graph, under the additional condition that $L<\eta_1$. After such drastic reduction of the number of possible paths, in § 4.5 we complete the proof of Theorem 1.2, with arguments similar in spirit to the analysis given by Cusick and Flahive in § 1 of [CuFl] for the classical Lagrange spectrum.

In § 5 we prove Theorem 1.1. In § 5.1 we introduce a lexicographic order on half-infinite words in the letters $a,b$, where $a=1,4,2,4$ and $b=1,3$ and the induced order on the values of the function $L^\sigma:\Xi_0\to\mathbb{R}_+$. Since we believe that the set $\mathbb{K}$ of such values is a Cantor set, we present the proof as an iterative construction of what we believe are the first two levels of a Cantor set. The first level is given in § 5.2 and the second level is given in § 5.3. Finally in § 5.4, by rephrasing the previous results, we conclude the proof Theorem 1.1.

## 2. A formula in terms of the action of $\mathrm{SL}(2,\mathbb{Z})$

2.1. **Continued fraction and** $\mathrm{SL}(2,\mathbb{Z})$**.** Consider an irrational number $\alpha=a_0+[a_1,a_2,\dots]$, where $a_0$ is the integer part of $\alpha$ and the sequence $(a_n)_{n\in\mathbb{N}^*}$ of positive integers corresponds to the fractional part. For any $n\geq 1$ we set

$$[a_1,\dots,a_n]:=\cfrac{1}{a_1+\cfrac{1}{a_2+\dots+\cfrac{1}{a_n}}}.$$

The $n$-th *Gauss approximation* of $\alpha$ is $p_n/q_n:=a_0+[a_1,a_2,\dots,a_n]$. For any such $n$, the *intermediate Farey approximations* are

$$\frac{p_{n,i}}{q_{n,i}}:=a_0+[a_1,a_2,\dots,a_{n-1},i]\text{ with }1\leq i<a_n.$$

Consider the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}$ by homographies, that is

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\alpha:=\frac{a\alpha+b}{c\alpha+d}.\tag{2.1}$$

Consider the elements

$$T:=\begin{pmatrix}1&1\\0&1\end{pmatrix},\ V:=\begin{pmatrix}1&0\\1&1\end{pmatrix}\text{ and }R:=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

and recall that $\{T,V\}$ are a set of generators. Thus, also $\{T,R\}$ generate $\mathrm{SL}(2,\mathbb{Z})$, by observing that

$$V=R\circ T^{-1}\circ R^{-1}.$$

The following Lemma holds both for Gauss and Farey approximations. For simplicity we state it just for the former.

**Lemma 2.1.** *If* $\alpha=a_0+[a_1,a_2,\dots]$ *then the sequence of co-slopes* $p_n/q_n$ *of the* Gauss approximations *of* $\alpha$ *is given by*

$$p_n/q_n=T^{a_0}\circ V^{a_1}\circ\dots\circ V^{a_{n-1}}\circ T^{a_n}\cdot 0,\qquad\textit{for even } n;$$
$$p_n/q_n=T^{a_0}\circ V^{a_1}\circ\dots\circ T^{a_{n-1}}\circ V^{a_n}\cdot\infty,\qquad\textit{for odd } n.$$

*Proof.* Just recall that the recursive relations satisfied by the Gauss approximations show that the sequence $(p_n, q_n)$ is obtained by setting $(p_{-2}, q_{-2}) = (0, 1)$ and $(p_{-1}, q_{-1}) = (1, 0)$ and then applying for any $k \in \mathbb{N}$ the recursive relations

$$\begin{pmatrix} p_{2k-1} & p_{2k} \\ q_{2k-1} & q_{2k} \end{pmatrix} = \begin{pmatrix} p_{2k-1} & p_{2k-2} \\ q_{2k-1} & q_{2k-2} \end{pmatrix} \circ T^{a_{2k}} \text{ and } \begin{pmatrix} p_{2k+1} & p_{2k} \\ q_{2k+1} & q_{2k} \end{pmatrix} = \begin{pmatrix} p_{2k-1} & p_{2k} \\ q_{2k-1} & q_{2k} \end{pmatrix} \circ V^{a_{2k+1}}.$$

□

2.2. **Action of** $\mathrm{SL}(2,\mathbb{Z})$ **on reduced origamis.** The *Veech group* of a translation surface $S$ is the subgroup $\mathrm{SL}(S)$ of $\mathrm{SL}(2,\mathbb{R})$ of those $G$ such that $G \cdot S = S$. It is easy to see that $\mathrm{SL}(S)$ is never co-compact and it is also well-known that the quotient $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(S)$ has finite volume if and only if the orbit $\mathrm{SL}(2,\mathbb{R}) \cdot S$ is closed in its stratum (this was stated by Veech in [Ve], a proof can be found in § 5 of [SmWe]). This last property is always true for origamis.

We say that a square-tiled surface $S$ is *reduced* if $\langle \mathrm{Hol}(S) \rangle = \mathbb{Z}^2$, that is the set of (relative) periods of $S$ generates $\mathbb{Z}^2$ as subgroup of $\mathbb{R}^2$. This implies that the Veech group $\mathrm{SL}(S)$ of $S$ is a finite-index subgroup of $\mathrm{SL}(2,\mathbb{Z})$. For any origami $S$ the Veech group $\mathrm{SL}(S)$ and $\mathrm{SL}(2,\mathbb{Z})$ share a common subgroup of finite index (see [HuLe]).

The action of $\mathrm{SL}(2,\mathbb{R})$ on translation surfaces induces an action of $\mathrm{SL}(2,\mathbb{Z})$ on square tiled surfaces. If $S$ is any square tiled surface, denote by $\mathcal{O}(S)$ its orbit under $\mathrm{SL}(2,\mathbb{Z})$, that is

$$\mathcal{O}(S) := \{Y = A \cdot S \; ; \; A \in \mathrm{SL}(2,\mathbb{Z})\}$$

The number of squares $N$ of an origami $S$ is obviously preserved under the action of $\mathrm{SL}(2,\mathbb{Z})$.

**Lemma 2.2** (Hubert-Lelièvre, [HuLe])**.** *The* $\mathrm{SL}(2,\mathbb{Z})$*-orbit* $\mathcal{O}(S)$ *of a reduced square-tiled surface* $S$ *with* $N$ *squares is the set of reduced square-tiled surfaces with* $N$ *squares in its* $\mathrm{SL}(2,\mathbb{R})$*-orbit.*

According to Lemma 2.2 above, the set of reduced origamis is preserved under the action of $\mathrm{SL}(2,\mathbb{Z})$, moreover each orbit $\mathcal{O}(S)$ is finite and we have the identification

$$\mathcal{O}(S) = \mathrm{SL}(2,\mathbb{Z})/\mathrm{SL}(S).$$

The action of $\mathrm{SL}(2,\mathbb{Z})$ passes to the quotient $\mathcal{O}(S)$. We chose the generators $\{T, R\}$ for $\mathrm{SL}(2,\mathbb{Z})$. In terms of $T$ and $R$, this action is represented by a oriented graph $\mathcal{G}(S)$ whose vertices are the elements of $\mathcal{O}(S)$ and whose oriented edges correspond to the operations $Y \mapsto T \cdot Y$ and $Y \mapsto R \cdot Y$ for $Y \in \mathcal{O}(S)$. Orbits of reduced origami in $\mathcal{H}(2)$ with prime number of squares $N$ have been classified in [HuLe], according to the number of integer *Weierstrass points*, then the classification was extended to the case of non prime $N$ in [Mc]. When $N = 3$ or when $N$ is even, there exists only one orbit. For any odd $N \geq 5$ there are two orbits. The first, called AN, contains reduced origamis with only one integer Weierstrass point. The second, called BN, contains reduced origamis with exactly three integer Weierstrass points. We refer to [HuLe] and [Mc] for more details. The complete description of the Orbit B7 is given in § 3.1, which is all we need in this paper.

2.2.1. *Cusps.* Let $S$ be a reduced origami. The cusps of $\mathbb{H}/\mathrm{SL}(S)$ correspond to conjugacy classes under $\mathrm{SL}(S)$ of its *primitive parabolic elements*, that is the elements in $\mathrm{SL}(S)$ with trace equal to $\pm 2$, where primitive means not powers of other parabolic elements of $\mathrm{SL}(S)$. If $S$ is a reduced origami the eigendirections of parabolic elements of $\mathrm{SL}(S)$ are exactly the elements of $\mathbb{Q}$. Therefore the cusps of $\mathbb{H}/\mathrm{SL}(S)$ correspond to equivalence classes for the homographic action $p/q \mapsto A \cdot p/q$ of $\mathrm{SL}(S)$ on $\mathbb{Q}$. Lemma 2.3 below gives a representation of cusps in terms of the action of $\mathrm{SL}(2,\mathbb{Z})$ (a proof can be found in [HuLe]).

**Lemma 2.3** (Zorich)**.** *Let* $S$ *be a reduced origami. Then the cusps of* $\mathbb{H}/\mathrm{SL}(S)$ *are in bijection with the* $T$*-orbits in* $\mathcal{O}(S)$*.*

2.3. **Multiplicity of a rational direction.** Fix a reduced square-tiled surface $S$ and denote by $\rho : S \to \mathbb{T}^2$ the ramified covering onto the standard torus. If $\gamma : [0,1] \to S$ is a saddle connection for $S$, we define its *multiplicity* $m(\gamma)$ as the degree of the map $t \mapsto \rho \circ \gamma(t)$. The *multiplicity of a rational direction with co-slope $p/q$ over the surface $S$* is the minimal multiplicity among all saddle connections on $S$ with the same co-slope $p/q$, that is the number $m(p/q; S)$ defined by

$$m(p/q;S) := \min\{m(\gamma); \quad \gamma \text{ saddle connection with } \operatorname{Hol}(\gamma) \wedge (p+iq) = 0\}. \tag{2.2}$$

The multiplicity is *covariant* under the left action of $\mathrm{SL}(2,\mathbb{Z})$, that is

$$m(p/q;S) = m(A \cdot p/q; A \cdot S). \tag{2.3}$$

**Remark 2.4.** *If the co-slopes $p/q$ and $p'/q'$ are in the same cusp equivalence class, then there exists some $A \in \mathrm{SL}(S)$ such that $p'/q' = A \cdot (p/q)$. Thus Equation* (2.3) *implies*

$$m(p/q;S) = m(p'/q';S).$$

Fix $n \in \mathbb{N}$ and consider positive integers $a_1, \dots, a_n$. Define the element $g(a_1,\dots,a_n)$ of $\mathrm{SL}(2,\mathbb{Z})$ by

$$g(a_1,\dots,a_n) := (T^{-a_n}R)\dots(T^{-a_2}R)(T^{a_1}R), \quad \text{ if } n \text{ is even };$$
$$g(a_1,\dots,a_n) := (T^{a_n}R)\dots(T^{-a_2}R)(T^{a_1}R), \quad \text{ if } n \text{ is odd }.$$

**Lemma 2.5.** *For any finite sequence $a_1,\dots,a_n$ we have*

$$m([a_1,\dots,a_n];S) = m(\infty; R \cdot g(a_1,\dots,a_n) \cdot S).$$

*Proof.* Recall that projectively we have $R^2 = \mathrm{Id}$, that is $R = R^{-1}$. Observe that $\infty = R \cdot 0$. According to Lemma 2.1 any rational number $[a_1,\dots,a_n]$ in $(0,1)$ can be written as

$$[a_1,\dots,a_n] = g(a_1,\dots,a_n)^{-1} \cdot R \cdot \infty.$$

Write for simplicity $g := g(a_1,\dots,a_n)$. The Lemma follows from the covariance of multiplicity stated in Equation (2.3), since we have that

$$m([a_1,\dots,a_n];S) = m(g^{-1}\cdot R \cdot \infty; S) = m(g^{-1} \cdot R \cdot \infty; g^{-1} \cdot R^2 \cdot g \cdot S) = m(\infty; R \cdot g \cdot S).$$

□

2.4. **Selection of relevant rational approximations.** Given $\alpha \in (0,1)$ with continued fraction expansion $\alpha = [a_1,a_2,\dots]$, recall that for any $n$ and any $i$ with $1 \le i < a_n$ *Gauss approximations* and *Farey approximations* were defined respectively by

$$\frac{p_n}{q_n} := [a_1,a_2,\dots,a_n] \text{ and } \frac{p_{n,i}}{q_{n,i}} := [a_1,a_2,\dots,a_{n-1},i].$$

The following result, which follows from classical continued fractions properties, is key for the renormalization formula. We provide a proof for completeness.

**Lemma 2.6.** *Fix $\alpha = [a_1,a_2,\dots]$. For any $n$ we have*

$$\frac{1}{q_n \cdot |q_n\alpha - p_n|} = [a_n,\dots,a_1] + a_{n+1} + [a_{n+2},a_{n+3},\dots]$$

*Moreover, for any $n$ and any $i$ with $1 \le i < a_n$ we have*

$$\frac{1}{q_{n,i} \cdot |q_{n,i}\alpha - p_{n,i}|} = [i,\dots,a_1] + [a_n - i, a_{n+1},\dots].$$

*Note:* For any $n$ and any $i$ with $1 \le i < a_n$ we have $[i,\dots,a_1] + [a_n - i, a_{n+1},\dots] < 2$, which corresponds to the well-known fact that $q|q\alpha - p| < 1/2$ just for $p/q = p_n/q_n$.

*Proof.* Suppose that $(p, q)$ and $(p', q')$ form a basis of $\mathbb{Z}^2$, so that $qp' - pq' = \pm 1$, and assume also that $(\alpha, 1)$ belongs to the convex cone spanned by these two vectors, that is $(q\alpha - p)(q'\alpha - p') < 0$. Then we have

$$\frac{1}{q|q\alpha - p|} = \left|\frac{qp' - pq' + qq'\alpha - qq'\alpha}{q(q\alpha - p)}\right| = \frac{q'}{q} + \frac{p' - q'\alpha}{q\alpha - p}.$$

In order to prove both the two parts of the statement we set $q' := q_{n-1}$ and $p' := p_{n-1}$. Observe that for any $n$ and any $i$ with $1 \leq i \leq a_n$, thus both for Farey and Gauss approximations, we have

$$\frac{q_{n-1}}{q_{n,i}} = [i, a_{n-1}, \dots, a_1].$$

To simplify the notation, for any $n$ and any $i$ set $l_{n,i} := |q_{n,i}\alpha - p_{n,i}|$. Set also $l_n := |q_n\alpha - p_n|$. The first part of the statement follows observing that $l_{n-1} = a_{n+1} \cdot l_n + l_{n+1}$, so that

$$\frac{l_{n-1}}{l_n} = a_{n+1} + [a_{n+2}, a_{n+2}, \dots].$$

The second part of the statement follows observing that $l_{n,i} = l_n + (a_n - i) \cdot l_{n-1}$, so that

$$\frac{l_{n-1}}{l_{n,i}} = \frac{1}{a_n - i + \frac{l_n}{l_{n-1}}} = [a_n - i, a_{n+1}, \dots].$$

□

**Lemma 2.7.** *Let $\alpha$ be an irrational slope. For any $\epsilon > 0$ there exists $Q > 0$ such that for any rational $p/q$ with $q > Q$ and which is neither a Gauss approximation of $\alpha$ nor a Farey approximation we have*

$$q \cdot |q\alpha - p| \geq 1 + 2\left(\frac{1}{L(\mathbb{T}^2, \alpha)} - \epsilon\right).$$

*Proof.* We assume $p/q < \alpha$, the other case being the same, then consider $n$ corresponding to those Gauss approximations such that $p_n/q_n < \alpha < p_{n-1}/q_{n-1}$. Assume also $a_n \geq 2$, the lattice argument for the case $a_n = 1$ being the same. According to the assumption in the statement, let $p_{n,i}/q_{n,i}$ and $p_{n,i+1}/q_{n,i+1}$ be two consecutive Farey approximations of $\alpha$ such that we have the strict inequality

$$p_{n,i}/q_{n,i} < p/q < p_{n,i+1}/q_{n,i+1} < \alpha.$$

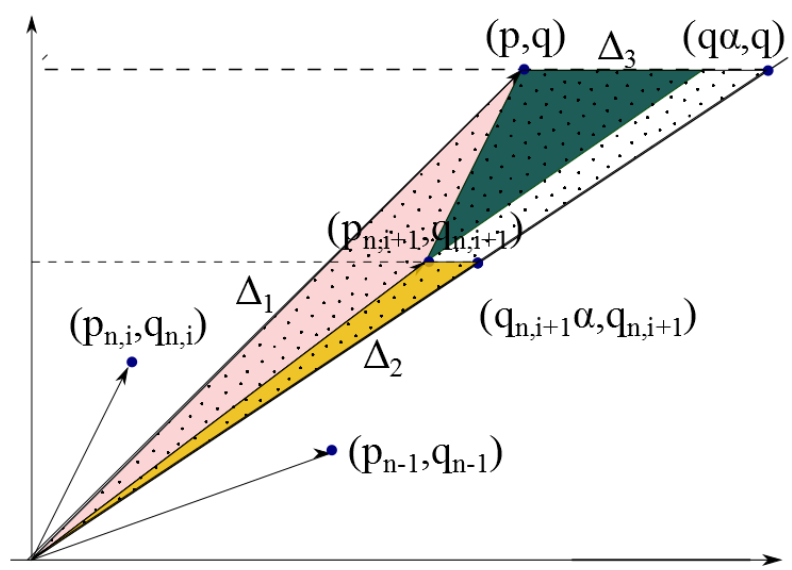


FIGURE 1. Illustration for the proof of Lemma 2.7.

We have $(p_{n,i+1}, q_{n,i+1}) = (p_{n,i}, q_{n,i}) + (p_{n-1}, q_{n-1})$. Moreover the integer vectors $(p_{n-1}, q_{n-1})$ and $(p_{n,i}, q_{n,i})$ form a basis of $\mathbb{Z}^2$, therefore we have a decomposition

$$(p, q) = x(p_{n-1}, q_{n-1}) + y(p_{n,i}, q_{n,i}) \text{ with } x \geq 1 \text{ and } y \geq 2.$$

Condition $y \geq 2$ holds because the linear combinations with $y = 1$ correspond to Farey's approximations. One can see that the triangle with vertices $(p, q)$, $(q\alpha, q)$ and $(0, 0)$ (which is dotted in Figure 1) contains the union of the triangles $\Delta_1$, $\Delta_2$ and $\Delta_3$ (see Figure 1), which are disjoint in the interior, where $\Delta_1$ is the triangle with vertices $(p, q)$, $(p_{n,i+1}, q_{n,i+1})$ and $(0, 0)$, $\Delta_2$ is the triangle with vertices $(p_{n,i+1}, q_{n,i+1})$, $(q_{n,i+1}\alpha, q_{n,i+1})$ and $(0, 0)$ and $\Delta_3$ is the triangle obtained translating by the vector $(p_{n,i+1}, q_{n,i+1})$ the triangle with vertices $(p_{n,i}, q_{n,i})$, $(q_{n,i}\alpha, q_{n,i})$ and $(0, 0)$. One can see that the area of the big triangle (dotted in Figure) is $q|q\alpha - p|$ and by computing the areas $A(\Delta_i)$ of the triangles one has that $A(\Delta_1) = |qp_{n,i+1} - q_{n,i+1}p|$, $A(\Delta_2) = |q_{n,i+1}|q_{n,i+1}\alpha - p_{n,i+1}|$ and $A(\Delta_3) = |q_{n,i}|q_{n,i}\alpha - p_{n,i}|$. Therefore we have

$$q|q\alpha - p| \geq |qp_{n,i+1} - q_{n,i+1}p| + q_{n,i}|q_{n,i}\alpha - p_{n,i}| + q_{n,i+1}|q_{n,i+1}\alpha - p_{n,i+1}|.$$

Fix $\epsilon > 0$ and set $a := (L(\mathbb{T}^2, \alpha))^{-1}$. If $n$ is big enough then we have both $q_{n,i}|q_{n,i}\alpha - p_{n,i}| > a - \epsilon$ and $q_{n,i+1}|q_{i+1}\alpha - p_{i+1}| > a - \epsilon$, hence $q|q\alpha - p| > 1 + 2(a - \epsilon)$. □

2.5. **The formula with continued fraction.** Fix a reduced origami $S$; let $N_S$ be the number of squares of $S$ and let

$$M_S := \max_{p/q \in \mathbb{Q}} m(p/q; S).$$

Consider irrational slopes $\alpha = [a_1, a_2, \dots]$ in $(0, 1)$. Recall that the function $L(\mathbb{T}^2, \alpha)$ is invariant under the homographic action of $\mathrm{SL}(2, \mathbb{Z})$. More generally, the function $\alpha \mapsto L(S, \alpha)$ is invariant under the Veech group $\mathrm{SL}(S)$ of $S$. Since the latter acts expansively and transitively on $\mathbb{R} \cup \{\infty\}$, then in order to compute $\mathcal{L}(S)$ it is enough to consider $\alpha \in (0, 1)$, that is

$$\mathcal{L}(S) = \{L(S, \alpha); 0 < \alpha < 1\}.$$

The following result was proved in [HuMarUl].

**Lemma 2.8** (Lemma 5.10 in [HuMarUl]). *Consider a reduced origami $S$. We have*

$$L(S, \alpha) := N_S \cdot \limsup_{q,p \to \infty} \frac{1}{m^2(p/q, S) \cdot q \cdot |q\alpha - p|}.$$

For any positive integer $n$ and any integer $i$ with $1 \leq i \leq a_n$ set

$$D(n, i, \alpha) := [a_n, \dots, a_1] + a_{n+1} + [a_{n+2}, a_{n+3}, \dots] \text{ if } i = a_n$$
$$D(n, i, \alpha) := [i, \dots, a_1] + [a_n - i, a_{n+1}, \dots] \text{ if } 1 \leq i < a_n.$$

**Theorem 2.9.** *Let $S$ be a reduced origami and let $\alpha = [a_1, a_2, \dots] \in (0, 1)$ be an irrational slope such that*

$$L(\mathbb{T}^2, \alpha) > M_S^2 - 2.$$

*Then we have*

$$L(S, \alpha) = N_S \cdot \limsup_{n \to \infty} \max_{1 \leq i \leq a_n} \frac{D(n, i, \alpha)}{m^2(\infty; R \cdot g(a_1, \dots, a_{n-1}, i) \cdot S)}.$$

**Remark 2.10.** *Theorem 2.9 generalizes the nice classical formula*

$$L(\mathbb{T}^2, \alpha) := \limsup_{n \to \infty} ([a_n, \dots, a_1] + a_{n+1} + [a_{n+2}, a_{n+3}, \dots])$$

*(remark that $D(n, i, \alpha)$ coincides with the numerator above for $i = a_n$). In* [HuMarUl] *we also prove a similar generalization of the classical formula to any square tiled surface (see Theorem 5.12 in* [HuMarUl]*), but which uses only Gauss approximations and holds in particular for $L(\mathbb{T}^2, \alpha) > M_S^2$. By considering also Farey approximations, the above formula allows to compute all values with $L(\mathbb{T}^2, \alpha) > M_S^2 - 2$.*

*Proof of Theorem 2.9.* Set $a := L(\mathbb{T}^2, \alpha)^{-1}$ and observe that the assumption in the statement is equivalent to $(M_S^2 - 2) \cdot a < 1$. Therefore consider $\epsilon > 0$ such that

$$1 + 2(a - \epsilon) > M_S^2 \cdot (a + \epsilon).$$

For any rational $p/q$ with $q$ big enough and which is neither a Farey nor a Gauss approximation of $\alpha$ then Lemma 2.7 implies

$$q \cdot |q\alpha - p| > 1 + 2(a - \epsilon).$$

On the other hand there exists infinitely many $n$ such that $q_n \cdot |q_n\alpha - p_n| < a + \epsilon$, therefore we have

$$\liminf_{n\to\infty} m^2(p_n/q_n; S) \cdot q_n \cdot |q_n\alpha - p_n| < M_S^2(a + \epsilon).$$

It follows that $\liminf_{q,p\to\infty} m^2(p/q; S) \cdot q \cdot |q\alpha - p|$ is taken either along the sequence of Farey approximations, or along the subsequence of Gauss approximation. The Theorem follows from the formulae in Lemma 2.6 and Lemma 2.7. □

## 3. The formula for the Orbit B7

3.1. **Description of the Orbit B7 in $\mathcal{H}(2)$.** Let $\mathcal{O}$ be the B-orbit of reduced square tiled surfaces with 7 squares. The orbit contains 36 elements, partitioned into 8 cusps (see § 2.2.1). Denote by the cusps by the letters $A, B, C, D, E, F, G, H$. For each cusp $X$ denote by $w = w(X)$ its width, which coincides with the number of square-tiled surfaces in the cusp equivalence class. We have

$$w(A) = w(B) = w(C) = 7 \ , \ w(D) = w(G) = w(H) = 3 \ , \ w(E) = 5 \text{ and } w(F) = 1.$$

Denote by $X_j$ the elements in the cusp $X$, where the index $j$ is an integer with $0 \leq j \leq w(X) - 1$, so that for example the elements of cusp $C$ are $C_0$, $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, $C_6$. The surfaces in $\mathcal{O}$ are drawn in figure 2. In the Figure, each surface corresponds to one of the polygons with specific glueings in the boundary. In each polygon, opposite vertical segments are always identified. For $F_0$, opposite horizontal sides are identified. For the surfaces $A_0$, $B_0$, $D_0$, $E_0$, $G_0$ and $H_0$ the identifications of horizontal sides are simply determined by length of sides. For $C_0$ the identification of horizontal sides is the only respecting length of sides and giving genus $g = 2$. Finally, for surfaces $X_j$ with $j > 0$ identifications of horizontal sides are those induced by the corresponding identifications on horizontal sides of $X_0$ and the action of $T^j$.

Denote by $\mathcal{G}$ the directed graph whose vertices are the elements of $\mathcal{O}$ and with labeled arrows for the action of $T$ and $R$, as in § 2.2. The graph $\mathcal{G}$ is shown in figure 3, where the oriented arrows outside the circle represent the action of $T$, while the arrows inside the circle represent the action of $R$ and are unoriented since $R = R^{-1}$.

Define a function $m : \mathcal{O} \to \{1, 2\}$ on the elements of $\mathcal{O}$ by setting $m(X) := m(\infty, X)$ for $X \in \mathcal{O}$, where $m(\infty, \cdot)$ is the multiplicity of the horizontal direction (recall the definition given in Equation (2.2)). Then, one can verify from Figure 2 that we have

$$m(C_j) = 2 \text{ for } j = 0, 1, \ldots, 6.$$
$$m(X_j) = 1 \text{ if } X_j \notin \{C_i; 0 \leq i \leq 6\}.$$

**Remark 3.1.** *According to Lemma 2.5, all the possible values of the multiplicity function*

$$m : \mathbb{Q} \times \mathcal{O} \to \mathbb{N}^* \ , \ (p/q, X_j) \mapsto m(p/q, X_j)$$

*are obtained by its restriction to $\{\infty\} \times \mathcal{O}$. The co-slope $p/q = \infty$ corresponds to the horizontal direction, and it is evident from Figure 2 that $m(\infty, X_j) \in \{1, 2\}$ for any $X_j \in \mathcal{O}$, since all horizontal saddle connections cross either one or at most two squares. It follows that for any $X_j \in \mathcal{O}$ we have*

$$M_S = \max\{m(p/q, X_j), p/q \in \mathbb{Q}\} = 2.$$

*On the other hand for any $\alpha$ irrational we have*

$$L(\mathbb{T}^2, \alpha) \geq \sqrt{5} > 2 = M_S^2 - 2,$$

*therefore the formula for $L(X_i, \alpha)$ in Theorem 2.9 holds for any $\alpha$ irrational. This is the reason for choosing the Orbit B.*

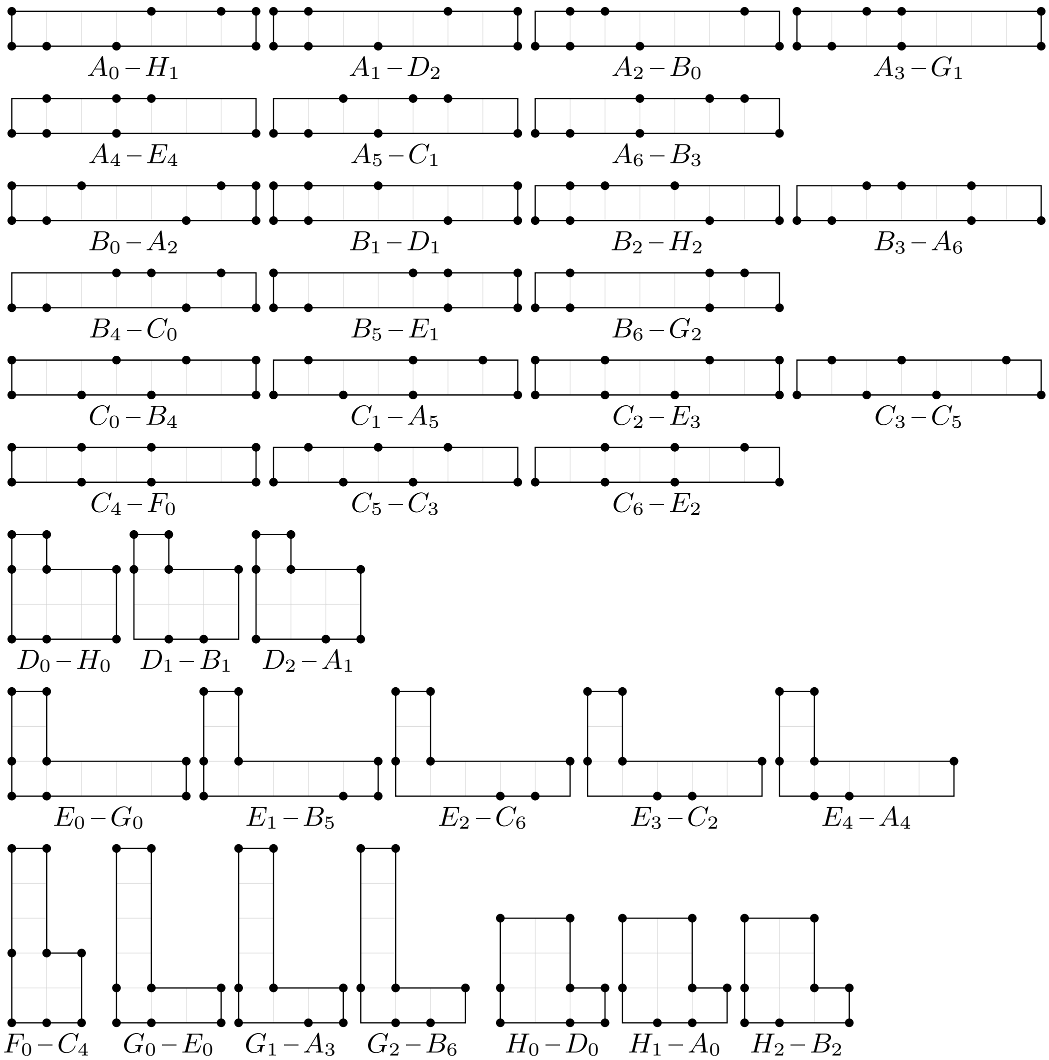


FIGURE 2. Surfaces in orbit $\mathcal{O}$, by cusp. Opposite vertical segments are always identified. In $F_0$ opposite horizontal sides are also identified. In cusps $A$, $B$, $D$, $E$, $G$ and $H$ identification of horizontal sides are simply determined by their length. In cusp $C$ one also has to impose genus $g = 2$ for the quotient surface. Under each surface, its name and the name of its image under $R$. Action of $T$ corresponds to moving to the right.

Fix $X_j$ and an irrational number $\alpha = [a_0, a_1, \dots]$. The continued fraction expansion of $\alpha$ gives a sequence of elements $g(a_1, \dots, a_{n-1}, i)$ in $\mathrm{SL}(2, \mathbb{Z})$, which are defined in § 2.3, where $n$ is a positive integer and the integer $i$ satisfies $1 \leq i \leq a_n$. They produce a path in $\mathcal{O}$, denoted by $(X_j, \alpha)$, whose vertices are the elements

$$g(a_1, \dots, a_{n-1}, i) \cdot X_j,$$

spanned in order as $n$ grows and, for each fixed $n$, the index $i$ grows in the range $1 \leq i \leq a_n$. More precisely, if $\alpha = [a_1, a_2, \dots]$ is the continued fraction expansion of $\alpha$, then the first $a_1$ vertices of the path $(X_j, \alpha)$ in $\mathcal{O}$ are

$$TR \cdot X_j, T^2 R \cdot X_j, \dots, T^{a_1} R \cdot X_j,$$

then the second $a_2$ vertices are

$$T^{-1} R(T^{a_1} R \cdot X_j), T^{-2} R(T^{a_1} R \cdot X_j), \dots, T^{-a_2} R(T^{a_1} R \cdot X_j)$$

and so on. We stress that at each iteration we *alternate the sign* of the powers of $T$ used, namely we use alternatively $T$ or $T^{-1}$. The vertices reached just before an arrow of type $R$, i.e. the ones of

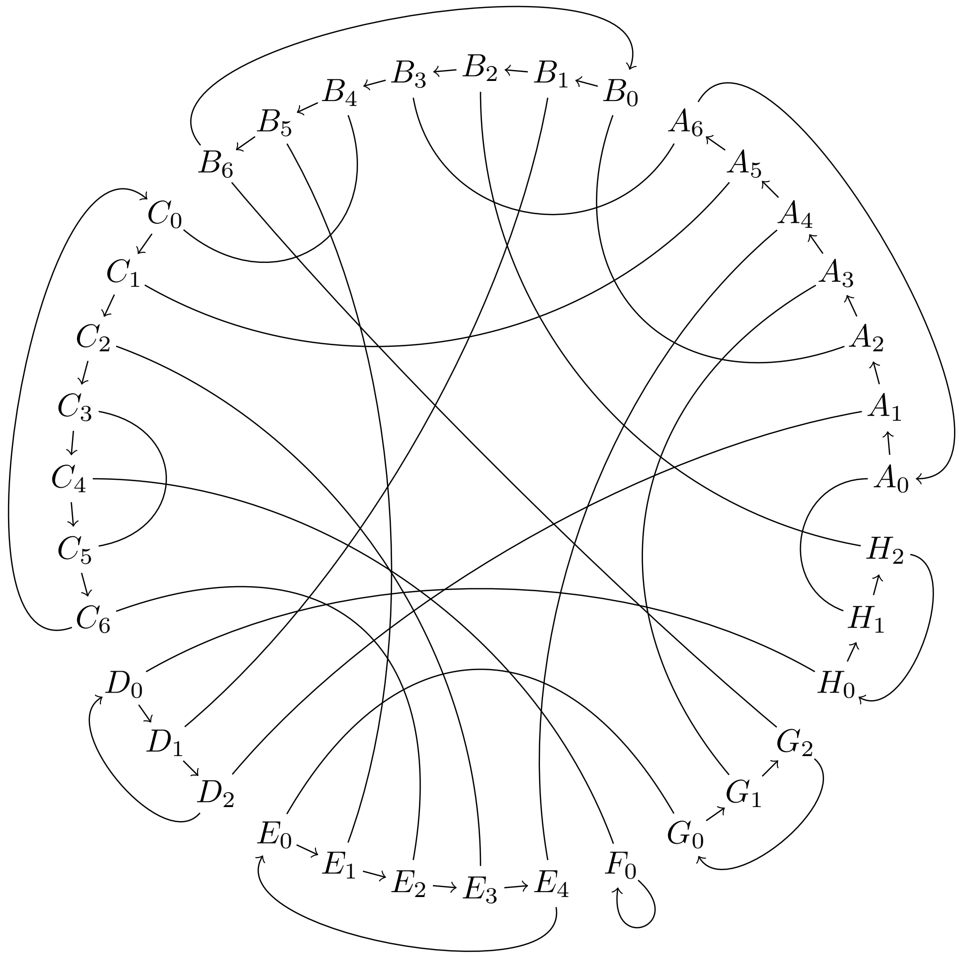


FIGURE 3. The graph $\mathcal{G}$ for orbit $\mathcal{O}$, with arrows along and outside the circle for the action of $T$, and unoriented edges inside the circle for the (involutive) action of $R$.

the form $g(a_1, \ldots, a_n) \cdot X_j$, correspond to *Gauss approximations*. The other vertices are of the form $g(a_1, \ldots, a_{n-1}, i) \cdot X_j$ for some intermediate $1 \leq i < a_n$ and correspond to intermediate *Farey approximations*.

Define the support $\mathrm{Supp}(X_j, \alpha)$ of the path $(X_j, \alpha)$ as the set of those $Y_k \in \mathcal{O}$ such that there exists infinitely many $n$ with $Y_k = g(a_1, \ldots, a_n) \cdot X_j$. Observe that the definition uses Gauss approximations $[a_1, \ldots, a_n]$ of $\alpha$ and not Farey approximations, thus elements of the form $Y_k = g(a_1, \ldots, a_{n-1}, i) \cdot X_j$ for infinitely many $n$ and infinitely many $i$ with $1 \leq j < a_n$ may *not* belong to $\mathrm{Supp}(X_j, \alpha)$ (they only do if they appear infinitely often *also* as Gauss approximations).

3.2. **The formula with continued fraction for the Orbit B7.** Recall that any reduced origami $X_j$ in the Orbit B7 has $N = 7$ squares. As we explained in Remark 3.1, the formula for $L(X_i, \alpha)$ in Theorem 2.9 holds for any $\alpha = [a_0, a_1, \ldots]$ irrational.

**Corollary 3.2.** *Let $X_j$ be any element in the Orbit B7 in $\mathcal{H}(2)$. Then for any irrational number $\alpha$ we have*

$$L(X_j, \alpha) = 7 \cdot \limsup_{n \to \infty} \max_{1 \leq i \leq a_n} \frac{D(n, i, \alpha)}{m^2\big(R \cdot g(a_1, \ldots, a_{n-1}, i) \cdot X_j\big)}. \tag{3.1}$$

3.3. **Periodic elements.** Let us recall that Lagrange spectra of translation surfaces are the closure of values corresponding to periodic orbits (see Theorem 1.5 in [HuMarUl]), which in the case of origamis are given by (co)slopes which are quadratic irrationals and hence define a periodic path on the graph associated to the $\mathrm{SL}(2,\mathbb{R})$ orbit. Let us hence explain how to compute Lagrange values associated to periodic paths.

An *even loop* on $\mathcal{O}$ is the datum $(X_j,\alpha)$, where $X_j\in\mathcal{O}$ and $\alpha$ is a quadratic irrational $\alpha=[\overline{a_1,\dots,a_{2N}}]$ whose period $a_1,\dots,a_{2N}$ has $2N$ entries, such that

$$g(a_1,\dots,a_{2N})\cdot X_j=X_j.$$

Let $\alpha=[\overline{a_1,\dots,a_{2N}}]$ be a quadratic irrational. The following is easy to prove.

(1) For any $n$ with $1\le n\le 2N$ we have
$$D(n,a_n,\alpha)=[\overline{a_n,\dots,a_1,a_{2N},\dots,a_{n+1}}]+a_{n+1}+[\overline{a_{n+2},\dots,a_{2N},a_1,\dots,a_{n+1}}]$$
where the cyclic the order of the $2N$ entries $a_1,\dots,a_{2N}$ is inverted in the first summand and it is preserved in the third.

(2) For any $n$ with $1\le n\le 2N$ such that $a_n\ge 2$ and any $i$ with $1\le i\le a_n-1$ we have
$$D(n,i,\alpha)=[i,\overline{a_{n-1},\dots,a_1,a_{2N},\dots,a_n}]+[a_n-i,\overline{a_{n+1},\dots,a_{2N},a_1,\dots,a_n}]$$
where the cyclic order of the entries $a_1,\dots,a_{2N}$ is inverted in the period of the first summand the and it is preserved in the period of the second.

For example, consider a quadratic irrational of the form $\alpha=[\overline{a_1,a_2,a_3,a_4,a_5,a_6}]$, where $2N=6$. For $n=3$ and $n=5$ we have respectively

$$D(3,a_3,\alpha):=[\overline{a_3,a_2,a_1,a_6,a_5,a_4}]+a_4+[\overline{a_5,a_6,a_1,a_2,a_3,a_4}]$$
$$D(5,a_5,\alpha):=[\overline{a_5,a_4,a_3,a_2,a_1,a_6}]+a_6+[\overline{a_1,a_2,a_3,a_4,a_5,a_6}].$$

Moreover, for $n=3$, assuming that $a_3\ge 2$, for any $i$ with $1\le i\le a_3-1$ we have

$$D(3,i,\alpha):=[i,\overline{a_2,a_1,a_6,a_5,a_4,a_3}]+[a_3-i,\overline{a_4,a_5,a_6,a_1,a_2,a_3}].$$

For an even loop $(X_j,\alpha)$ where $\alpha=[\overline{a_1,\dots,a_{2N}}]$ is a quadratic irrational whose period $a_1,\dots,a_{2N}$ has $2N$ entries Equation (3.1) becomes

$$L(X_i,\alpha)=7\cdot\max_{1\le n\le 2N}\max_{1\le i\le a_n}\frac{D(n,i,\alpha)}{m^2\big(R\cdot g(a_1,\dots,a_{n-1},i)\cdot X_j\big)}. \tag{3.2}$$

Let us compute the values associated to two special even loops. The first will play the role of the golden mean, i.e. it gives the smallest value in the Lagrange spectrum of the B7 Orbit. The second computation is presented as an extra example of application of Formula (3.2).

**Lemma 3.3.** *The data $(C_6,[\overline{1,3}])$ and $(C_3,[\overline{5,2}])$ define even loops in $\mathcal{O}$ and we have*
$$L(C_6,[\overline{1,3}])=\phi_1=7+14\cdot[\overline{3,1}].$$
$$L(C_3,[\overline{5,2}])=14\cdot[1,\overline{5,2}]=11,832159\pm 10^{-6}.$$

*Proof.* One can verify that both $(C_6,[\overline{1,3}])$ and $(C_3,[\overline{5,2}])$ are even loops with $2N=2$, which before closing up go respectively through the vertices $E_3,C_1,C_0$ and through the vertices $C_6,C_0,C_1,C_2,C_3,C_4$ (recall that the vertices of the graph after an $R$ arrow are *not* recorded in the loop). Moreover, to simplify the notation, we divide everything by a factor 7. To compute $L(C_6,[\overline{1,3}])$ we apply Equation (3.2) and take the maximum among the values below

$$\frac{D(1,1,[\overline{1,3}])}{m^2(R\cdot g(1)\cdot C_6)}=\frac{[\overline{1,3}]+3+[\overline{1,3}]}{m^2(C_2)}=\frac{[\overline{1,3}]+3+[\overline{1,3}]}{4}=1,1456435\pm 10^{-6}$$
$$\frac{D(2,1,[\overline{1,3}])}{m^2(R\cdot g(1,1)\cdot C_6)}=\frac{[1,\overline{1,3}]+[2,\overline{1,3}]}{m^2(A_5)}=\frac{[1,\overline{1,3}]+[2,\overline{1,3}]}{1}=0,916515\pm 10^{-6}$$
$$\frac{D(2,2,[\overline{1,3}])}{m^2(R\cdot g(1,2)\cdot C_6)}=\frac{[2,\overline{1,3}]+[1,\overline{1,3}]}{m^2(B_4)}=\frac{[2,\overline{1,3}]+[1,\overline{1,3}]}{1}=0,916515\pm 10^{-6}$$
$$\frac{D(2,3,[\overline{1,3}])}{m^2(R\cdot g(1,3)\cdot C_6)}=\frac{[\overline{1,3}]+1+[\overline{1,3}]}{m^2(E_2)}=\frac{[\overline{3,1}]+1+[\overline{3,1}]}{1}=1,527524\pm 10^{-6}.$$

The first part is proved recalling that $\phi_1 = 7(1+[\overline{1,3}])$. To compute $L(C_3,[\overline{5,2}])$ we apply Equation (3.2) and take the maximum among the values below

$$
\begin{aligned}
\frac{D(1,1,[\overline{5,2}],C_3)}{m^2(R\cdot g(1)\cdot C_3)} &= \frac{[1,\overline{2,5}]+[4,\overline{2,5}]}{m^2(E_2)} = \frac{[1,\overline{2,5}]+[4,\overline{2,5}]}{1} = 0,910165 \pm 10^{-6}\\
\frac{D(1,2,[\overline{5,2}],C_3)}{m^2(R\cdot g(2)\cdot C_3)} &= \frac{[2,\overline{2,5}]+[3,\overline{2,5}]}{m^2(B_4)} = \frac{[2,\overline{2,5}]+[3,\overline{2,5}]}{1} = 0,696009 \pm 10^{-6}\\
\frac{D(1,3,[\overline{5,2}],C_3)}{m^2(R\cdot g(3)\cdot C_3)} &= \frac{[3,\overline{2,5}]+[2,\overline{2,5}]}{m^2(A_5)} = \frac{[2,\overline{2,5}]+[3,\overline{2,5}]}{1} = 0,696009 \pm 10^{-6}\\
\frac{D(1,4,[\overline{5,2}],C_3)}{m^2(R\cdot g(4)\cdot C_3)} &= \frac{[4,\overline{2,5}]+[1,\overline{2,5}]}{m^2(E_3)} = \frac{[4,\overline{2,5}]+[1,\overline{2,5}]}{1} = 0,910165 \pm 10^{-6}\\
\frac{D(1,5,[\overline{5,2}],C_3)}{m^2(R\cdot g(5)\cdot C_3)} &= \frac{[\overline{5,2}]+2+[\overline{5,2}]}{m^2(C_5)} = \frac{2+2[\overline{5,2}]}{4} = 0,591607 \pm 10^{-6}\\
\frac{D(2,1,[\overline{5,2}],C_3)}{m^2(R\cdot g(5,1)\cdot C_3)} &= \frac{[1,\overline{5,2}]+[1,\overline{5,2}]}{m^2(F_0)} = \frac{2[1,\overline{5,2}]}{1} = 1,690309 \pm 10^{-6}\\
\frac{D(2,2,[\overline{5,2}],C_3)}{m^2(R\cdot g(5,2)\cdot C_3)} &= \frac{[\overline{2,5}]+5+[\overline{2,5}]}{m^2(C_5)} = \frac{5+2[\overline{2,5}]}{4} = 1,479019 \pm 10^{-6}.
\end{aligned}
$$

The second is proved observing that $14\cdot[1,\overline{5,2}] = 7\cdot 1,690309 \pm 10^{-6}$. □

## 4. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. For $i=1,2,3$ consider the quadratic irrationals $\eta_i$ with $\phi_\infty < \eta_1 < \eta_2 < \eta_3$ defined by

$$
\begin{aligned}
\eta_1 &:= 7\cdot\frac{[1,4,2,\overline{1,5}]+5+[1,5,1,\overline{1,5}]}{4} = 11,655309 \pm 10^{-6}\\
\eta_2 &:= 7\cdot\frac{[1,\overline{1,6}]+6+[\overline{6,1}]}{4} = 11,688957 \pm 10^{-6}\\
\eta_3 &:= 7\cdot\left(1+[1,\overline{1,6}]+[\overline{6,1}]\right) = 11,755835 \pm 10^{-6}.
\end{aligned}
$$

In § 4.1 below we consider a graph $\mathcal{A}$, obtained from a subgraph of $\mathcal{G}$, that we call *accessible graph*. Then we introduce a graph $\mathcal{I}$ called *intermediate graph*, whose vertices are the vertices of $\mathcal{A}$ and whose arrows are some combinations of the arrows of $\mathcal{A}$, and finally we define a subgraph $\mathcal{S}$ of $\mathcal{I}$, called *small graph*. In Proposition 4.7 we show that data $X_j$ and $\alpha$ with $L(X_j,\alpha) < \eta_3$ satisfy $\mathrm{Supp}(X_j,\alpha)\subset\mathcal{A}$. Moreover, eventually (i.e. after dropping some finite initial segment) we have that $a_n \leq 6$ and the path $(X_j,\alpha)$ can be decomposed into elementary operations which are arrows of the intermediate graph $\mathcal{I}$. Then in Proposition 4.3 we show that if $L(X_j,\alpha) < \eta_1$ then the path $(X_j,\alpha)$ can be decomposed into elementary operations which are arrows of the small graph $\mathcal{S}$, and this is the main technical tool in the proof of Theorem 1.2. As an intermediate step towards the proof of Proposition 4.3, in Corollary 4.11 we prove that if $L(X_j,\alpha) < \eta_2$ then $\mathrm{Supp}(X_j,\alpha)\subset\mathcal{A}$ and $a_n \leq 5$ eventually.

4.1. **The intermediate graph $\mathcal{I}$ and the small graph $\mathcal{S}$.** For $X_j\in\mathcal{O}$ and a positive integer $a$, let $Y_k$ and $Z_l$ be the two elements in $\mathcal{O}$ such that we have respectively $T^aR(X_j) = Y_k$ and $T^{-a}R(X_j) = Z_l$. Represent the two elementary operations above as

$$X_j \xrightarrow{a+} Y_k \text{ and } X_j \xrightarrow{a-} Z_l.$$

The *accessible graph* $\mathcal{A}$ (shown to the right in Figure 4) is the graph whose vertices are $C_2$, $C_3$, $C_4$, $C_5$, $C_6$, $E_2$, $E_3$ and $F_0$ and whose elementary arrows are

$$\begin{aligned}
&T : C_i \to C_{i+1} \text{ for } i = 2,3,4,5\\
&T^3 : C_6 \to C_2\\
&T : F_0 \to F_0\\
&T : E_2 \to E_3\\
&R : E_2 \to C_6\ ,\ R : E_3 \to C_2\ ,\ R : C_4 \to F_0 \text{ and } R : C_3 \to C_5.
\end{aligned}$$

The accessible graph is induced from a proper subgraph of $\mathcal{G}$ as follows. In Figure 4, the accessible graph $\mathcal{A}$ is represented on the right side, whereas on the left side one can see a graph which is a proper subgraph of the graph $\mathcal{G}$: $\mathcal{A}$ is obtained from the subgraph of $\mathcal{G}$ by removing the vertices $C_0$ and $C_1$ and the arrows $T : C_6 \to C_0$, $T : C_0 \to C_1$ and $T : C_1 \to C_2$, then replacing them by the arrow $T^3 : C_6 \to C_2$. We stress that even if the elements $C_0$ and $C_1$ are not part of the accessible graph $\mathcal{A}$, paths $(X_j, \alpha)$ whose support is in $\mathcal{A}$ and which only use arrows in $\mathcal{A}$ pass through these two elements, but never at times corresponding to Gauss approximations, since no $R$ arrow leaves from this vertices.

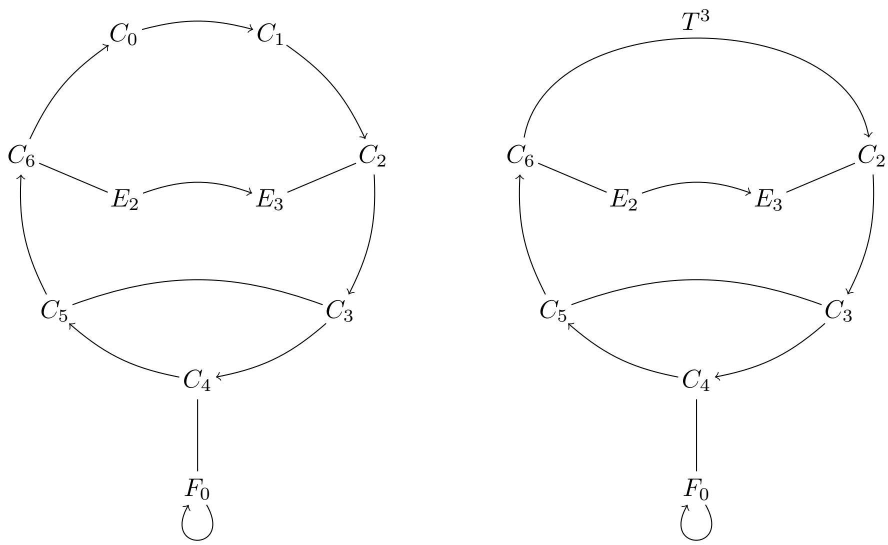


FIGURE 4. Removing the vertices $C_0$ and $C_1$ and the arrows $T : C_6 \to C_0$, $T : C_0 \to C_1$ and $T : C_1 \to C_2$ from the graph on the left of the picture, which is a subgraph of $\mathcal{G}$, and replacing them by the arrow $T^3 : C_6 \to C_2$, one gets the accessible graph $\mathcal{A}$, represented on the right of the picture, which contains all paths that can contribute to the bottom of the Lagrange spectrum.

We say that an elementary operation $X_j \overset{a\pm}{\longrightarrow} Y_k$ is *admissible in* $\mathcal{A}$ if it appears infinitely many times in a path $(X_j, \alpha)$ with support in $\mathcal{A}$. This is a non-trivial obstruction, due to the combinatorics of the graph $\mathcal{A}$ and to the fact that concatenation of elementary operations must respect the sign alternation, in order to define a path $\big(g(a_1, \ldots, a_n) \cdot X_j\big)_{n\geq 1}$ as described in § 3.1. For example, as one can see in Figure 4, no elementary operation of the form $E_3 \overset{a+}{\longrightarrow} Y_k$ is admissible in $\mathcal{A}$, because the only elementary operation ending in $E_3$ is $C_6 \overset{1+}{\longrightarrow} E_3$ and the former cannot follow in concatenation the latter, otherwise the sign alternation would not be respected. Roughly speaking, one can see that if an elementary operation $X_j \overset{a\pm}{\longrightarrow} Y_k$ in $\mathcal{A}$ can can be concatenated to an other elementary operation in $\mathcal{A}$ both in the past and in the future, then it admits arbitrarily long past and future continuations, and thus is admissible in $\mathcal{A}$.

Let us now introduce the *intermediate graph* $\mathcal{I}$, which is represented in Figure 5 and is the graph induced from $\mathcal{A}$ taking some admissible elementary operations (see Lemma 4.1 below). The vertices of the intermediate graph $\mathcal{I}$ are $C_2$, $C_3$, $C_4$, $C_5$, $C_6$, $E_2$, $E_3$ and $F_0$. The arrows in $\mathcal{I}$ correspond to the 30 elementary operations listed here below, where for convenience of notation we also introduce a short name for each, which is either $\mathcal{F}_\xi$ or $\mathcal{G}_\xi$, or a primed version of the same names, i.e. $\mathcal{F}'_\xi$ or $\mathcal{G}'_\xi$, and where $\xi$ is a label. Primes reflect a symmetry of the graph under the involution $\psi$ defined below, primed arrows are shown in a lighter shade of the corresponding unprimed version in Figure 5. The label $\xi$ associated to an elementary operation $X_j \xrightarrow{a\pm} Y_k$ is a contraction (preserving the injectivity of the labeling) of the three-symbols data $XaY$, representing the cusps $X$ and $Y$ respectively of the starting point $X_j$ and the ending point $Y_k$ of the elementary operation, and the number $a$ of iterates of $T$ or of $T^{-1}$ that it contains. The elementary operations named by $\mathcal{F}_\xi$ are the ones which will survive in a further reduction.

$$
\begin{array}{lll}
\mathcal{F}_1 := C_6 \xrightarrow{1+} E_3 & \text{and} & \mathcal{F}'_1 := C_2 \xrightarrow{1-} E_2 \\
\mathcal{F}_2 := C_5 \xrightarrow{2+} C_5 & \text{and} & \mathcal{F}'_2 := C_3 \xrightarrow{2-} C_3 \\
\mathcal{F}_3 := E_3 \xrightarrow{3-} C_6 & \text{and} & \mathcal{F}'_3 := E_2 \xrightarrow{3+} C_2 \\
\mathcal{F}_{C4} := C_5 \xrightarrow{4-} C_6 & \text{and} & \mathcal{F}'_{C4} := C_3 \xrightarrow{4+} C_2 \\
\mathcal{F}_{E4} := E_3 \xrightarrow{4-} C_5 & \text{and} & \mathcal{F}'_{E4} := E_2 \xrightarrow{4+} C_3
\end{array}
$$

$$
\begin{array}{lll}
\mathcal{G}_{CC} := C_5 \xrightarrow{1+} C_4 & \text{and} & \mathcal{G}'_{CC} := C_3 \xrightarrow{1-} C_4 \\
\mathcal{G}_{CF} := C_4 \xrightarrow{1+} F_0 & \text{and} & \mathcal{G}'_{CF} := C_4 \xrightarrow{1-} F_0 \\
\mathcal{G}_{FC} := F_0 \xrightarrow{1+} C_5 & \text{and} & \mathcal{G}'_{FC} := F_0 \xrightarrow{1-} C_3 \\
\mathcal{G}_{E5} := E_3 \xrightarrow{5-} C_4 & \text{and} & \mathcal{G}'_{E5} := E_2 \xrightarrow{5+} C_4 \\
\mathcal{G}_{C5} := C_5 \xrightarrow{5-} C_5 & \text{and} & \mathcal{G}'_{C5} := C_3 \xrightarrow{5+} C_3 \\
\mathcal{G}_{5C} := F_0 \xrightarrow{5-} C_6 & \text{and} & \mathcal{G}'_{5C} := F_0 \xrightarrow{5+} C_2 \\
\mathcal{G}_{6C} := C_5 \xrightarrow{6-} C_4 & \text{and} & \mathcal{G}'_{6C} := C_3 \xrightarrow{6+} C_4 \\
\mathcal{G}_{F6} := F_0 \xrightarrow{6-} C_5 & \text{and} & \mathcal{G}'_{6C} := F_0 \xrightarrow{6+} C_3 \\
\mathcal{G}_{E6} := E_3 \xrightarrow{6-} C_3 & \text{and} & \mathcal{G}'_{E6} := E_2 \xrightarrow{6+} C_5 \\
\mathcal{G}_{C6} := C_5 \xrightarrow{6+} C_2 & \text{and} & \mathcal{G}'_{C6} := C_3 \xrightarrow{6-} C_6.
\end{array}
$$

**Lemma 4.1.** *The elementary operations $X_j \xrightarrow{a\pm} Y_k$ with $1 \leq a \leq 6$ which are admissible in $\mathcal{A}$ are exactly the 30 elementary arrows of the intermediate graph $\mathcal{I}$, together with the* 10 *elementary operations $C_4 \xrightarrow{a\pm} F_0$ with $2 \leq a \leq 6$.*

*Proof.* The proof is a direct verification. For any vertex $X_j$ of the accessible graph $\mathcal{A}$ we consider all the admissible elementary operations as in the statement which start at $X_j$. For example $C_6 \xrightarrow{1+} E_3$ is the only elementary operation starting at $C_6$. Then compatibility with concatenation implies that $E_3 \xrightarrow{a-} C_{9-a}$ with $a = 3, 4, 5, 6$ are the only admissible operations starting at $E_3$, where $a = 1, 2$ are not allowed because $C_0$ and $C_1$ do not belong to $\mathcal{A}$. Details are left to the reader. $\square$

The graph $\mathcal{I}$ has an internal symmetry which we will exploit to further reduce the paths to consider to study the bottom of the spectrum. In Figure 5 this symmetry correspond to a reflection through a vertical axes passing through $C_4$ and $F_0$. Formally, we describe this symmetry by defining an involution $\psi : \mathcal{I} \to \mathcal{I}$ which acts on vertices by sending

$$\psi(C_2) = C_6 \ , \ \psi(C_3) = C_5 \ , \ \psi(E_3) = E_2 \ , \ \psi(C_4) = C_4 \ , \ \psi(F_0) = F_0.$$

One can easily verify that $\psi^2 = \mathrm{Id}$ and hence $\psi$ is indeed an involution. Extend the function $\psi$ to the set of paths on $\mathcal{S}$ by

$$
\begin{aligned}
&\psi(\mathcal{F}_\xi) := \mathcal{F}'_\xi \text{ for any label } \xi = 1, 2, 3, C4, E4 \\
&\psi(\mathcal{G}_\xi) := \mathcal{G}'_\xi \text{ for any label } \xi = CC, CF, FC, E5, C5, 5C, 6C, F6, E6, C6.
\end{aligned}
$$

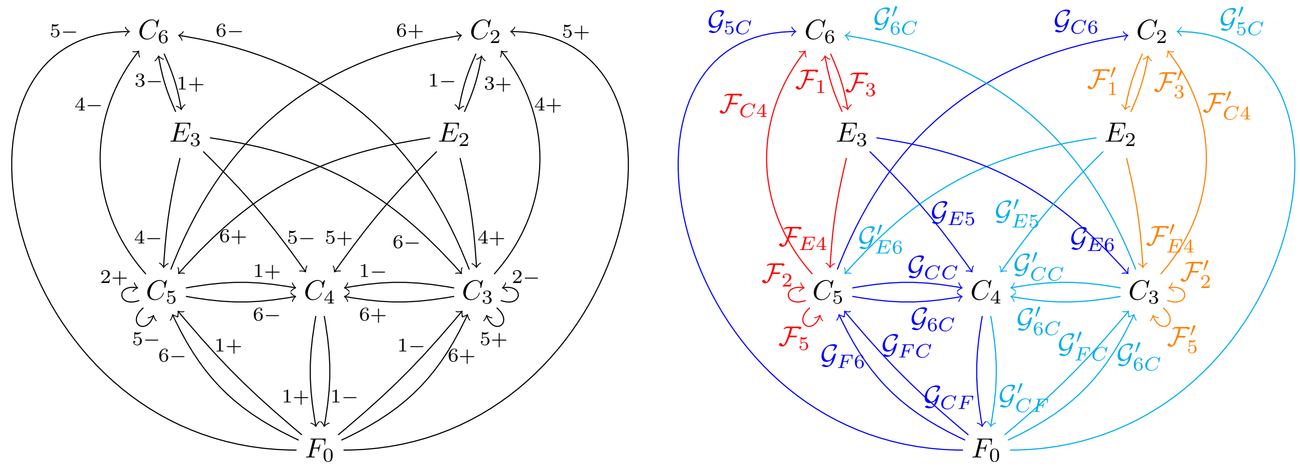


FIGURE 5. The elementary operations with are admissible in the accessible graph $\mathcal{A}$ are the arrows of the intermediate graph $\mathcal{I}$. In the left copy, these elementary operations are represented explicitly, whereas in the right copy they are represented by their short names. Colors are explained in the text.

**Remark 4.2.** *Observe that the multiplicity is preserved by $\psi$, that is $m(\psi(X_j)) = m(X_j)$ for any element $X_j \in \mathcal{I}$. Furthermore, notice that $\psi$ maps arrows labeled by $+a$ to $-a$ and conversely.*

If we consider only paths which eventually do not cross the vertex $F_0$ (and hence neither the vertex $C_4$), one can check that the remaining paths live in one of two connected components, mapped to each other by the involution. Let us hence define a graph which encodes (a subset of) the operations of one of these two connected components.

The *small graph* (show in Figure 7) is the subgraph $\mathcal{S}$ of $\mathcal{I}$ whose vertices are the elements $C_5$, $C_6$, $E_3$ and whose arrows are $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_{4C}, \mathcal{F}_{4E}$. Observe that the subgraph $\mathcal{I}'$ of $\mathcal{I}$ whose vertices are $C_2$, $C_3$, $C_5$, $C_6$, $E_2$, $E_3$ and whose arrows are $\mathcal{F}_\xi$ and $\mathcal{F}'_\xi$ for $\xi = 1, 2, 3, 4C, 4E$ has two connected components, one being the small graph $\mathcal{S}$ and the other being the image $\psi(\mathcal{S})$ of $\mathcal{S}$ under the involution $\psi$.

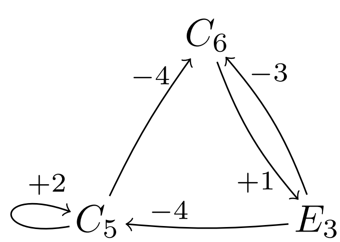


FIGURE 6. The *small graph* $\mathcal{S}$.

Proposition 4.3 below is the main technical result in the proof of Theorem 1.2. The proof of Proposition 4.3 is the subject of the next subsections §4.2, §4.3 and §4.4.

**Proposition 4.3.** *Consider $X_j \in \mathcal{O}$ and $\alpha$ such that*

$$L(X_j, \alpha) < \eta_1.$$

*Then, eventually, the only elementary operations appearing in the path $(X_j, \alpha)$ are:*

(1) *either arrows of the small graph $\mathcal{S}$, that belong to the list: $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_{4C}$, $\mathcal{F}_{4E}$;*
(2) *or arrows of the image $\psi(\mathcal{S})$ of the small graph, that is belong to the list: $\mathcal{F}'_1$, $\mathcal{F}'_2$, $\mathcal{F}'_3$, $\mathcal{F}'_{4C}$, $\mathcal{F}'_{4E}$.*

Lemma 4.4 below is a classical result on continued fraction playing a relevant role in several parts of this section. The proof is left to the reader.

**Lemma 4.4.** *Fix a positive integer $M$. Consider $m \in \mathbb{N}$ and positive integers $b_1, \dots, b_m$. Let $K$ be the Cantor set of those real numbers $x$ of the form*

$$x = [b_1, \dots, b_m, a_1, a_2, \dots] \text{ where } 1 \leq a_n \leq M \text{ for any } n,$$

*that is the set of those $x$ whose continued fraction expansion starts with the prescribed entries $b_1, \dots, b_m$ and then all the other entries $a_n$ satisfy $1 \leq a_n \leq M$. Then we have*

$$\min K = [b_1, \dots, b_m, \overline{M, 1}] \text{ and } \max K = [b_1, \dots, b_m, \overline{1, M}] \quad \text{if } m \text{ is even;}$$
$$\min K = [b_1, \dots, b_m, \overline{1, M}] \text{ and } \max K = [b_1, \dots, b_m, \overline{M, 1}] \quad \text{if } m \text{ is odd.}$$

*In particular, for the degenerate case $m = 0$, when there is no prescribed beginning for the continued fraction expansion of the elements of $K$, we have*

$$[\overline{M, 1}] = \min K \text{ and } [\overline{1, M}] = \max K.$$

4.2. **Reduction to the intermediate graph $\mathcal{I}$.** Observe that

$$\eta_3 < 7 \cdot \frac{7 + 2[\overline{7, 1}]}{4} = 12,693741 \pm 10^{-6}.$$

**Lemma 4.5.** *Consider $X_j \in \mathcal{O}$ and $\alpha = [a_1, a_2, \dots]$ irrational such that*

$$L(X_j, \alpha) < 7 \cdot \frac{7 + 2[\overline{7, 1}]}{4}.$$

*Then we eventually have $1 \leq a_n \leq 6$.*

*Proof.* Fix any $\delta > 0$. If the continued expansion of $\alpha$ contains infinitely many $a_n \geq 8$, then for any such $n$ which is big enough and for $i = a_n$ we have $D(n, i, \alpha) = D(n, a_n, \alpha) > 8$ and

$$L(X_j, \alpha) + \delta \geq 7 \cdot \frac{D(n, a_n, \alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} \geq 7 \cdot \frac{D(n, a_n, \alpha)}{4} > 14.$$

Since $\delta$ is arbitrary, we get $L(X_j, \alpha) \geq 14$, which is absurd. Now assume that the continued expansion of $\alpha$ contains infinitely many $a_n \geq 7$. For any such $n$ which is big enough, and for $i = a_n$, Lemma 4.4 implies $D(n, a_n, \alpha) > 7 + 2[\overline{7, 1}]$ and thus

$$L(X_j, \alpha) + \delta \geq \frac{D(n, a_n, \alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} \geq 7 \cdot \frac{7 + 2[\overline{7, 1}]}{4}.$$

Since $\delta$ is arbitrary, it follows $L(X_j, \alpha) \geq 7(7 + 2[\overline{7, 1}])/4$, which is absurd. □

**Lemma 4.6.** *Consider $X_j \in \mathcal{O}$ and $\alpha = [a_1, a_2, \dots]$ irrational such that $L(X_j, \alpha) < \eta_3$. Assume that there exists $Y_k \in \operatorname{Supp}(X_j, \alpha)$ with $m^2(R \cdot Y_k) = 1$. Then for any sufficiently large $n$ satisfying*

$$g(a_1, \dots, a_n) \cdot X_j = Y_k$$

*we have*

$$2 \leq a_n \leq 6 \ , \ a_{n+1} = 1 \ , \ 2 \leq a_{n+2} \leq 6$$
$$m^2(Rg(a_1, \dots, a_{n-1}) \cdot X_j) = m^2(Rg(a_1, \dots, a_{n+1}) \cdot X_j) = 4.$$

*Proof.* If $n$ is as in the statement, since $m^2(Rg(a_1, \dots, a_n) \cdot X_j) = 1$, then we have $a_{n+1} = 1$, otherwise Formula (3.1) would give

$$L(X_j, \alpha) + \delta \geq 7 \cdot \frac{D(n, a_n, \alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} > 7a_{n+1} \geq 14,$$

where $\delta > 0$ is an arbitrarily small positive constant, which is absurd. According to Lemma 4.5 we have $1 \leq a_n \leq 6$. Moreover, if either $a_n = 1$ or $a_{n+2} = 1$ then Lemma 4.4 implies

$$L(X_j, \alpha) + \delta \geq 7 \cdot \frac{D(n, a_n, \alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} > 7D(n, 1, \alpha) \geq 7(1 + [1, \overline{1, 6}] + [\overline{6, 1}])$$

for any arbitrarily small positive constant $\delta$ and any $n$ big enough, which is absurd. Since $a_n \geq 2$ and $a_{n+2} \geq 2$, it follows $m^2(Rg(a_1, \ldots, a_{n-1}) \cdot X_j) = m^2(Rg(a_1, \ldots, a_{n+1}) \cdot X_j) = 4$, otherwise, arguing as in the beginning of the proof one gets $L(X_j, \alpha) \geq 14$, which is absurd. □

**Proposition 4.7.** *Consider $X_j \in \mathcal{O}$ and $\alpha = [a_1, a_2, \ldots]$ irrational such that*

$$L(X_j, \alpha) < \eta_3.$$

*Then* $\mathrm{Supp}(X_j, \alpha)$ *is contained in the accessible graph $\mathcal{A}$, we have $1 \leq a_n \leq 6$ for any $n$ big enough and moreover the 30 arrows of the intermediate graph $\mathcal{I}$ are the only elementary operations that can appear infinitely many times in the path $(X_j, \alpha)$.*

*Proof.* Let $Y_k \in \mathrm{Supp}(X_j, \alpha)$ and consider $n$ with $g(a_1, \ldots, a_n) \cdot X_j = Y_k$. Assume that $n$ is large enough, so that Lemma 4.6 can be applied.

**Case 1.** We first assume

$$m^2(R \cdot Y_k) = 1$$

and we consider separately the two subcases $n$ even and $n$ odd.

**Case 1, even:** Assume that $n$ is even. According to Lemma 4.6 we have $a_{n+1} = 1$ and therefore

$$TR \cdot Y_k = g(a_1, \ldots, a_{n+1}) \cdot X_j.$$

The last equality, and again Lemma 4.6 imply $m^2(RTR \cdot Y_k) = 4$, which has for solutions

$$TR \cdot Y_k = A_5, B_4, C_3, C_5, E_2, E_3, F_0,$$

corresponding respectively to $Y_k = E_4, A_6, E_3, F_0, B_5, C_6, C_4$. In such list, the elements $Y_k$ satisfying $m^2(R \cdot Y_k) = 1$ are

$$Y_k = E_4, A_6, B_5, C_6, C_4.$$

Moreover, we also have the identity

$$T^{-a_n} Rg(a_1, \ldots, a_{n-1}) \cdot X_j = Y_k,$$

thus Lemma 4.6 implies also $m^2(T^{a_n} \cdot Y_k) = 4$. Independently on $a_n$, the only elements in the list $E_4, A_6, B_5, C_6, C_4$ satisfying this last condition are $Y_k = C_4$ and $Y_k = C_6$, which are both elements of the accessible graph $\mathcal{A}$.

**Case 1, odd:** Let us now assume that $n$ is odd, so that we have the identity

$$T^{-1} R \cdot Y_k = g(a_1, \ldots, a_{n+1}) \cdot X_j.$$

Lemma 4.6 and the identity above imply $m^2(RT^{-1}R \cdot Y_k) = 4$, which has for solutions

$$Y_k = B_3, E_1, F_0, E_2, C_2, A_4, C_4.$$

In such list, the elements $Y_k$ satisfying $m^2(R \cdot Y_k) = 1$ are $Y_k = B_3, E_1, C_2, A_4, C_4$. Moreover, we also have the identity

$$T^{a_n} Rg(a_1, \ldots, a_{n-1}) \cdot X_j = Y_k,$$

thus Lemma 4.6 implies also and $m^2(T^{-a_n} \cdot Y_k) = 4$. Independently on $a_n$, the only elements in the list $B_3, E_1, C_2, A_4, C_4$ satisfying this last condition are $Y_k = C_4$ and $Y_k = C_2$, which are both elements of the accessible graph $\mathcal{A}$.

**Case 2.** All other elements of $\mathcal{O}$ satisfy
$$m^2(R\cdot Y_k)=4$$
and they are $Y_k=A_5,B_4,C_3,C_5,E_2,E_3,F_0$. In this list, the only elements which are not in the accessible graph $\mathcal{A}$ are $Y_k=B_4$ and $Y_k=A_5$. If $B_4\in\mathrm{Supp}(X_j,\alpha)$ then there exist $i$ with $0\leq i\leq 6$ with $R\cdot B_i\in\mathrm{Supp}(X_j,\alpha)$. Since $m^2(R\cdot R\cdot B_i)=m^2(B_i)=1$ for any $i=0,\dots,6$, Lemma 4.6 (the part of the statement corresponding to $a_{n+1}=1$) implies
$$\begin{aligned}&\text{either } E_1=RT\cdot B_4=R\cdot B_5\in\mathrm{Supp}(X_j,\alpha)\\&\text{or } A_6=RT^{-1}\cdot B_4=R\cdot B_3\in\mathrm{Supp}(X_j,\alpha).\end{aligned}$$
The first case corresponds to $B_4=T^{-1}R\cdot E_1$ and Lemma 4.6 again implies $E_1=T^aR\cdot Z_l$ for some $a$ with $2\leq a\leq 6$ and some $Z_l\in\mathcal{O}$ with $m^2(R\cdot Z_l)=4$, and this is absurd since $m^2(T^{-a}\cdot E_1)=1$ for any $a$. The second case corresponds to $B_4=TR\cdot A_6$ and reasoning similarly we get necessarily $A_6=T^{-a}R\cdot Z_l$ for some $a$ with $2\leq a\leq 6$ and some $Z_l\in\mathcal{O}$ with $m^2(R\cdot Z_l)=4$, which is absurd because $m^2(T^a\cdot A_6)=1$ for any $a$. If $A_5\in\mathrm{Supp}(X_j,\alpha)$, arguing as for $B_4$, we get
$$\begin{aligned}&\text{either } B_3=RT\cdot A_5\in\mathrm{Supp}(X_j,\alpha)\\&\text{or } E_4=RT^{-1}\cdot A_5\in\mathrm{Supp}(X_j,\alpha).\end{aligned}$$
The first case corresponds to $A_5=T^{-1}R\cdot B_3$ and implies $B_3=T^aR\cdot Z_l$ for some $Z_l\in\mathcal{O}$ with $m^2(R\cdot Z_l)=4$, and this is impossible since $m^2(T^{-a}B_3)=1$ for any $a$. The second case corresponds to $A_5=TR\cdot E_4$ and implies $E_4=T^{-a}R\cdot Z_l$ for some $Z_l\in\mathcal{O}$ with $m^2(R\cdot Z_l)=4$, and this is impossible since $m^2(T^aE_4)=1$ for any $a$. This concludes the proof of this Case.

We proved that $\mathrm{Supp}(X_j,\alpha)$ is contained in the accessible graph $\mathcal{A}$. Moreover, according to Lemma 4.5 we eventually have $1\leq a_n\leq 6$. Finally, since $m^2(R\cdot C_4)=1$, then Lemma 4.6 implies that the elementary operations $C_4\xrightarrow{a\pm}F_0$ with $2\leq a\leq 6$ are forbidden. According to Lemma 4.1, it follows that the path $(X_j,\alpha)$ is eventually decomposed into elementary operations which are arrows of the intermediate graph $\mathcal{I}$. The Proposition is proved. □

4.3. **Blocks.** Proposition 4.7 is proved just considering elementary operations $X_j\xrightarrow{a\pm}Y_k$, that is segments of paths of length one. For a deeper analysis we need to consider finite segments with length bigger than one.

A *block* $H$ is a finite segment of a path $(X_j,\alpha)$ generated by some element $X_j\in\mathcal{O}$ and some $\alpha$ irrational. More precisely a block is a triple of data $H=(Y_k,\{b_1,\dots,b_m\},\epsilon)$ where $Y_k\in\mathcal{O}$, the entries $b_1,\dots,b_m$ are positive integers and $\epsilon\in\{+,-\}$. If $H$ is a block, there exists $X_j\in\mathcal{O}$ and $\alpha=[a_1,a_2,\dots]$ such that $Y_j\in\mathrm{Supp}(X_j,\alpha)$, that is $Y_k=g(a_1,\dots,a_n)\cdot X_j$ for infinitely many $n$, and moreover for any such $n$ we have $a_{n+1}=b_1,\dots,a_{n+m}=b_m$ and $\epsilon=+$ if $n$ is odd and $\epsilon=-$ if $n$ is even. In the following we will use often an alternative representation of blocks, with the explicit description of all operations as above. For example the blocks $H_1=(C_5,[5,6,1],-)$ and $H_2=(E_2,[3,1,6,5],+)$ will be represented respectively by the two sequences of operations
$$\begin{aligned}&H_1=C_5\xrightarrow{5-}C_5\xrightarrow{6+}C_2\xrightarrow{1-}E_2\\&H_2=E_2\xrightarrow{3+}C_2\xrightarrow{1-}E_2\xrightarrow{6+}C_5\xrightarrow{5-}C_4\end{aligned}$$
which appear as finite segments of the path $(E_2,\alpha=[\overline{3,1,6,5,6,1}])$. We will assume that blocks are contained in the accessible graph $\mathcal{A}$.

Let $H=(Y_k,\{b_1,\dots,b_m\},\epsilon)$ be a block which is contained in the accessible graph $\mathcal{A}$. We define
$$L^{inf}(H):=\inf L(X_j,\alpha),$$
where $(X_j,\alpha)$ varies among all the paths with $\mathrm{Supp}(X_j,\alpha)\subset\mathcal{A}$ which contains the block $H$ infinitely many times. If $M$ is a positive integer such that $b_i\leq M$ for all the entries of the block $H$ we define also $L^{inf}_M(H):=\inf L(X_j,\alpha)$ where $(X_j,\alpha)$ varies among all the paths with $\mathrm{Supp}(X_j,\alpha)\subset\mathcal{A}$ which contains the block $H$ infinitely many times and such that $\alpha=[a_1,a_2,\dots]$ eventually satisfies $a_n\leq M$. We have the following simple but very useful Lemma.

**Lemma 4.8.** *Fix a positive integer $M$ and consider a block $H = (Y_k, [b_1, \dots, b_m], \epsilon)$ of length $m \geq 1$ such that $b_i \leq M$ for any $i$ with $1 \leq i \leq m$. For any such $i$ we have*

$$L_M^{inf}(H) \geq 7 \cdot \frac{[b_i, \dots, b_1, \overline{c, d}] + b_{i+1} + [b_{i+2}, \dots, b_m, \overline{e, f}]}{m^2(R \cdot g(b_1, \dots, b_i) \cdot Y_k)},$$

*where*

$$\begin{aligned}
&(c,d) = (e,f) = (M,1) \text{ for } i \text{ even, } m \text{ odd}\\
&(c,d) = (e,f) = (1,M) \text{ for } i, m \text{ odd}\\
&(c,d) = (M,1) \text{ and } (e,f) = (1,M) \text{ for } i, m \text{ even}\\
&(c,d) = (1,M) \text{ and } (e,f) = (1,M) \text{ for } i \text{ odd, } m \text{ even.}
\end{aligned}$$

*Proof.* Let $(X_j, \alpha)$, with $\alpha = [a_1, a_2, \dots]$ be any path containing the block $H$ infinitely many times and such that eventually $a_n \leq M$. Fix any $\delta > 0$. Fix $i$ with $0 \leq i \leq m-1$. Formula (3.1) gives

$$L(X_j, \alpha) = 7 \limsup_{n \to \infty} \max_{1 \leq i \leq a_n} \frac{D(n, i, \alpha)}{m^2(R \cdot g(a_1, \dots, a_{n-1}, i) \cdot X_i)} \geq$$

$$7 \limsup_{n \to \infty} \frac{D(n, a_n, \alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_i)} = 7 \limsup_{n \to \infty} \frac{[a_n, a_{n-1}, \dots] + a_{n+1} + [a_{n+2}, a_{n+3}, \dots]}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_i)}.$$

The Lemma follows from Lemma 4.4 observing that, since $H$ appears infinitely often in the path generated by $(X_j, \alpha)$, then there is $n$ arbitrarily big with

$$a_{n+1-i} = b_1, \dots, a_{n+1} = b_{i+1}, \dots, a_{n+m-i} = b_m.$$

□

The following Lemma is also practical to reduce the number of cases in our analysis.

**Lemma 4.9.** *Fix a positive integer $M$ and consider a block $H = (Y_k, [b_1, \dots, b_m], \epsilon)$ of length $m \geq 1$ such that $b_i \leq M$ for any $i$ with $1 \leq i \leq m$. If $H$ is contained into the accessible graph $\mathcal{A}$, then we have*

$$L_M^{inf}(\psi(H)) = L_M^{inf}(H).$$

*Proof.* Consider a path $(X_j, \alpha)$ with $\mathrm{Supp}(X_j, \alpha) \subset \mathcal{A}$, containing the block $H$ infinitely many times and such that $\alpha = [a_1, a_2, \dots]$ eventually satisfies $a_n \leq M$. We can assume without loosing in generality that all elementary operations composing $(X_j, \alpha)$ are arrows of the intermediate graph. Thus one can define a new path $(X'_j, \alpha)$ by concatenation, in the same order, of the image under $\phi$ of the elementary operations composing $(X_j, \alpha)$, where obviously $\alpha = [a_1, a_2, \dots]$ keeps unchanged. The Lemma follows applying Formula (3.1) and recalling Remark 4.2. □

**Lemma 4.10.** *The inequalities below hold.*

$$\begin{aligned}
L_6^{inf}(F_0 \xrightarrow{6+} C_3) &= L_6^{inf}(F_0 \xrightarrow{6-} C_5) \geq \eta_2.\\
L_6^{inf}(C_3 \xrightarrow{6+} C_4) &= L_6^{inf}(C_5 \xrightarrow{6-} C_4) \geq \eta_2.\\
L_6^{inf}(C_5 \xrightarrow{6+} C_2) &= L_6^{inf}(C_3 \xrightarrow{6-} C_6) \geq \eta_2.\\
L_6^{inf}(E_2 \xrightarrow{6+} C_5) &= L_6^{inf}(E_3 \xrightarrow{6-} C_3) \geq \eta_2.
\end{aligned}$$

*Proof.* The four equalities for the eight blocks in the statement follow from Lemma 4.9 and from the correspondence between these blocks under the involution $\psi$, that is

$$\begin{aligned}
F_0 \xrightarrow{6-} C_5 &= \psi(F_0 \xrightarrow{6+} C_3)\\
C_5 \xrightarrow{6-} C_4 &= \psi(C_3 \xrightarrow{6+} C_4)\\
C_3 \xrightarrow{6-} C_6 &= \psi(C_5 \xrightarrow{6+} C_2)\\
E_3 \xrightarrow{6-} C_3 &= \psi(E_2 \xrightarrow{6+} C_5).
\end{aligned}$$

and by Remark 4.2, that implies that the desired values of $L_6^{inf}$ are the same.

The lemma thus follows proving that $L_6^{inf}(H_i) \geq \eta_2$ for $i = 1, \dots, 4$, where we set for simplicity

$$H_1 := F_0 \xrightarrow{6+} C_3 \;,\; H_2 := C_3 \xrightarrow{6+} C_4 \;,\; H_3 := C_5 \xrightarrow{6+} C_2 \;,\; H_4 := E_2 \xrightarrow{6+} C_5.$$

The block $H_1$ has as prolongation

$$H_1' := C_4 \xrightarrow{1-} F_0 \xrightarrow{6+} C_3,$$

Lemma 4.8 with parameters $m = 2$, $i = 2$ implies

$$L^{inf}(H_1') \geq 7 \cdot \frac{[1, \overline{1, 6}] + 6 + [\overline{6, 1}]}{4} = \eta_2.$$

Since $m(C_4) = 1$, then $L^{inf}(C_4 \xrightarrow{a-} F_0) > 7 \cdot 2$ for any $a \geq 2$ and therefore

$$L^{inf}(H_1) \geq \min\{14, L^{inf}(H_1')\} \geq \eta_2.$$

The block $H_2$ has as prolongation the block

$$H_2' := C_3 \xrightarrow{6+} C_4 \xrightarrow{1-} F_0,$$

Lemma 4.8 with parameters $m = 2$, $i = 1$ implies

$$L^{inf}(H_2') \geq 7 \cdot \frac{[1, \overline{1, 6}] + 6 + [\overline{6, 1}]}{4} = \eta_2.$$

Since $m(C_4) = 1$, then $L^{inf}(C_4 \xrightarrow{a-} F_0) > 7 \cdot 2$ for any $a \geq 2$ and therefore

$$L^{inf}(H_2) \geq \min\{14, L^{inf}(H_2')\} \geq \eta_2.$$

The block $H_3$ has as prolongation the block

$$H_3' := C_5 \xrightarrow{6+} C_2 \xrightarrow{1-} E_3,$$

Lemma 4.8 with parameters $m = 2$, $i = 2$ implies

$$L^{inf}(H_3') \geq 7 \cdot \frac{[1, \overline{1, 6}] + 6 + [\overline{6, 1}]}{4} = \eta_2.$$

Since $m(C_2) = 1$, then $L^{inf}(C_2 \xrightarrow{a-} X_j) > 7 \cdot 2$ for any $a \geq 2$ and therefore

$$L^{inf}(H_3) \geq \min\{14, L^{inf}(H')\} \geq \eta_2.$$

In fact one can also observe that $C_2 \xrightarrow{1-} E_3$ is the only arrow of the intermediate graph $\mathcal{I}$ which starts at $C_2$. Finally, the block $H_4$ has as prolongation the block

$$H_4' := C_5 \xrightarrow{6+} C_2 \xrightarrow{1-} E_3,$$

Lemma 4.8 with parameters $m = 2$, $i = 2$ implies

$$L^{inf}(H_4') \geq 7 \cdot \frac{[1, \overline{1, 6}] + 6 + [\overline{6, 1}]}{4} = \eta_2.$$

Since $m(C_2) = 1$, then $L^{inf}(C_2 \xrightarrow{a-} X_j) > 7 \cdot 2$ for any $a \geq 2$ and therefore

$$L^{inf}(H_4) \geq \min\{14, L^{inf}(H_4')\} \geq \eta_2.$$

As for $H_3$, one can also observe that $C_2 \xrightarrow{1-} E_3$ is the only arrow of the intermediate graph which starts at $C_2$. □

**Corollary 4.11.** *Let $(X_j, \alpha)$ be pair with $X_j \in \mathcal{O}$ and $\alpha = [a_1, a_2, \dots]$ is irrational such that*

$$L(X_j, \alpha) < \eta_2.$$

*Then* $\mathrm{Supp}(X_j, \alpha) \subset \mathcal{A}$*, we have $1 \leq a_n \leq 5$ for any big enough and the only elementary operations that can appear infinitely many times in $(X_j, \alpha)$ are the 22 arrows $X_j \xrightarrow{a\pm} Y_k$ of the intermediate graph $\mathcal{I}$ with $1 \leq a \leq 5$.*

*Proof.* Let $X_j \in \mathcal{O}$ and $\alpha = [a_1, a_2, \dots]$ be data as in the statement. Proposition 4.7 implies

$$\mathrm{Supp}(X_j, \alpha) \subset \mathcal{A}$$

and $a_n \leq 6$ for any $n$ big enough, moreover the only elementary operation that can appear infinitely many times in $(X_j, \alpha)$ are the 30 arrows of the intermediate graph $\mathcal{I}$. In particular, since $a_n \leq 6$ definetly, for any block $H$ appearing infinitely many times in the path $(X_j, \alpha)$ we have $L^{inf}(H) = L^{inf}_6(H)$. According to Lemma 4.10, none of the eight blocks listed in the Lemma appear infinitely many times in the path $(X_j, \alpha)$. Therefore, if there exists infinitely many $n$ with $a_n = 6$ then either the block $C_4 \xrightarrow{6+} F_0$ or the block $C_4 \xrightarrow{6-} F_0$ appear infinitely many times in $(X_j, \alpha)$, but this is absurd because $(R \cdot C_4) = 1$ and thus $L^{inf}(C_4 \xrightarrow{6+} F_0) > 7 \cdot 6$ and $L^{inf}(C_4 \xrightarrow{6-} F_0) > 7 \cdot 6$. □

4.4. **Proof of Proposition 4.3.** Consider data $X_j \in \mathcal{O}$ and $\alpha = [a_1, a_2, \dots]$ such that

$$L(X_j, \alpha) < \eta_1 = 7 \cdot \frac{[1,4,2,\overline{1,5}] + 5 + [1,5,1,\overline{1,5}]}{4} = 11,655309 \pm 10^{-6}.$$

Since $\eta_1 < \eta_2$, Corollary 4.11 implies that $\mathrm{Supp}(X_j, \alpha)$ is contained in the accessible graph $\mathcal{A}$ and $a_n \leq 5$ eventually. The only elementary operations ending in $F_0$ are $C_4 \xrightarrow{1+} F_0$ and $C_4 \xrightarrow{1-} F_0$, that is the arrows $\mathcal{G}_{CF}$ and $\mathcal{G}'_{CF}$ of the intermediate graph $\mathcal{I}$. We will now show that such elementary operations cannot be repeated infinitely many times in paths $(X_j, \alpha)$ with $L(X_j, \alpha) < \eta_1$.

**Lemma 4.12.** *We have*

$$L^{inf}_5(C_4 \xrightarrow{1+} F_0) = L^{inf}_5(C_4 \xrightarrow{1-} F_0) \geq \eta_1.$$

*Proof.* The equality in the statement holds because the two blocks correspond under the involution $\psi$, that is

$$C_4 \xrightarrow{1-} F_0 = \psi(C_4 \xrightarrow{1+} F_0).$$

The only arrow of the intermediate graph $\mathcal{I}$ which starts at $C_2$ is $C_2 \xrightarrow{1-} E_2$, therefore the elementary operation $F_0 \xrightarrow{2-} C_2$ is not admissible on the accessible graph $\mathcal{A}$. Moreover $C_2 \xrightarrow{1-} E_2$ is also the only arrow of $\mathcal{I}$ ending in $E_2$, therefore the elementary operation $E_2 \xrightarrow{2-} C_4$ is not admissible in the accessible graph $\mathcal{A}$. It follows that the only length-two blocks of the accessible graph $\mathcal{A}$ which start with $C_4 \xrightarrow{1+} F_0$ are

$$H_1 := C_4 \xrightarrow{1+} F_0 \xrightarrow{1-} C_3$$
$$H_2 := C_4 \xrightarrow{1+} F_0 \xrightarrow{5-} C_6.$$

For the same reason, the only length-two blocks of the accessible graph $\mathcal{A}$ which end with $C_4 \xrightarrow{1+} F_0$ are

$$H_3 := C_3 \xrightarrow{1-} C_4 \xrightarrow{1+} F_0$$
$$H_4 := E_3 \xrightarrow{5-} C_4 \xrightarrow{1+} F_0.$$

According to Lemma 4.6, we have $L^{inf}(H_1) \geq \eta_3 > \eta_1$ and $L^{inf}(H_3) \geq \eta_3 > \eta_1$. Therefore, if the path $(X_j, \alpha)$ contain infinitely many times $F_0$, then, eventually, at any occurrence of $F_0$ it has to contain the block

$$E_3 \xrightarrow{5-} C_4 \xrightarrow{1+} F_0 \xrightarrow{5-} C_6.$$

Moreover $C_6 \xrightarrow{1+} E_3$ is both the only arrow of $\mathcal{I}$ starting at $C_6$ and the only arrow in $\mathcal{I}$ ending in $E_3$. Therefore, at any occurrence of $F_0$ in the path $(X_j, \alpha)$ we must eventually have the block

$$C_6 \xrightarrow{1+} E_3 \xrightarrow{5-} C_4 \xrightarrow{1+} F_0 \xrightarrow{5-} C_6 \xrightarrow{1+} E_3.$$

Any path containing infinitely many times the block above must contain infinitely many time one of the blocks below

$$H'_1 := F_0 \xrightarrow{1+} C_5 \xrightarrow{4-} C_6 \xrightarrow{1+} E_3 \xrightarrow{5-} C_4 \xrightarrow{1+} F_0 \xrightarrow{5-} C_6 \xrightarrow{1+} E_3$$
$$H'_2 := C_5 \xrightarrow{2+} C_5 \xrightarrow{4-} C_6 \xrightarrow{1+} E_3 \xrightarrow{5-} C_4 \xrightarrow{1+} F_0 \xrightarrow{5-} C_6 \xrightarrow{1+} E_3$$
$$H'_3 := F_0 \xrightarrow{5-} C_6 \xrightarrow{1+} E_3 \xrightarrow{5-} C_4 \xrightarrow{1+} F_0 \xrightarrow{5-} C_6 \xrightarrow{1+} E_3.$$

The Lemma follows observing that

$$L_5^{inf}(H_3') \geq L_5^{inf}(H_1') \geq L_5^{inf}(H_2') \geq 7 \cdot \frac{[1,4,2,\overline{1,5}] + 5 + [1,5,1,\overline{1,5}]}{4}.$$

□

**Corollary 4.13.** *Consider data $(X_j, \alpha)$ such that $L(X_j, \alpha) < \eta_1$. The arrows that can appear infinitely many times in the path generated by $(X_j, \alpha)$ are either only the six arrows*

$$\mathcal{F}_1 := C_6 \xrightarrow{1+} E_3 \quad , \quad \mathcal{F}_2 := C_5 \xrightarrow{2+} C_5 \quad , \quad \mathcal{F}_3 := E_3 \xrightarrow{3-} C_6$$
$$\mathcal{F}_{C4} := C_5 \xrightarrow{4-} C_6 \quad , \quad \mathcal{F}_{E4} := E_3 \xrightarrow{4-} C_5 \quad , \quad \mathcal{G}_{C5} := C_5 \xrightarrow{5-} C_5$$

*or only their image under the involution $\phi$.*

*Proof.* According to Lemma 4.12, any arrow of the intermediate graph $\mathcal{I}$ which ends in $F_0$ cannot be contained infinitely many times in $(X_j, \alpha)$, thus $F_0$ does not belong to $\mathrm{Supp}(X_j, \alpha)$. But all arrows of the intermediate graph $\mathcal{I}$ starting at $C_4$ end in $F_0$, thus $C_4$ does not belong to $\mathrm{Supp}(X_j, \alpha)$. The statement follows recalling Lemma 4.10. □

According to Corollary 4.13, in order to finish the proof of Proposition 4.3, it is enough to exclude the elementary operation $C_5 \xrightarrow{5-} C_5$, or in other words the arrow $\mathcal{G}_{C5}$. This follows from the next Lemma.

**Lemma 4.14.** *We have*

$$L_5^{inf}(C_5 \xrightarrow{5-} C_5) \geq \eta_1$$

*Proof.* Observe first that

$$\eta_1 < 7 \cdot ([1,5,2,4,\overline{1,3}] + [1,4,\overline{1,3}]) = 11,706478 \pm 10^{-6}.$$

If $L_5^{inf}(C_5 \xrightarrow{5-} C_5) < \eta_1$ then there exists $X_j \in \mathcal{O}$ and $\alpha = [a_1, a_2, \dots]$ irrational with $1 \leq a_n \leq 5$ for any $n$ which generate a path containing $C_5 \xrightarrow{5-} C_5$ infinitely many times and such that $L^{inf}(X_j, \alpha) < \eta_1$. According to Corollary 4.13, the only elementary operations starting in $C_5$ that can appear infinitely many time in $(X_j, \alpha)$ are $C_5 \xrightarrow{5-} C_5$ and $C_5 \xrightarrow{4-} C_6$, and similarly the only elementary operations ending in $C_5$ that can appear infinitely many times in $(X_j, \alpha)$ are $C_5 \xrightarrow{2+} C_5$ and $E_3 \xrightarrow{4-} C_5$. Therefore, if the arrow $C_5 \xrightarrow{5-} C_5$ appears infinitely many time in the path generated by $(X_j, \alpha)$, then at any occurrence it also occurs the block

$$C_5 \xrightarrow{2+} C_5 \xrightarrow{5-} C_5 \xrightarrow{2+} C_5.$$

We apply Formula (3.1). Fix $\delta > 0$ and consider $N$ such that $g(a_1, \dots, a_{N-1}) \cdot X_j = C_5$ and $a_N = 2$, $a_{N+1} = 5$, $a_{N+2} = 2$ and moreover

$$L(X_j, \alpha) = 7 \cdot \limsup_{n \to \infty} \max_{1 \leq i \leq a_n} \frac{D(n,i,\alpha)}{m^2(R \cdot g(a_1, \dots, a_{n-1}, i) \cdot X_j)} >$$
$$7 \cdot \frac{D(N,1,\alpha)}{m^2(R \cdot g(a_1, \dots, a_{N-1}, 1) \cdot X_j)} + \delta$$

Since $g(a_1, \dots, a_{N-1}) \cdot X_j = C_5$, we have $g(a_1, \dots, a_{N-1}, 1) \cdot X_j = C_4$ and recalling that $m^2(R \cdot C_4) = 1$ we have

$$\frac{D(N,1,\alpha)}{m^2(R \cdot g(a_1, \dots, a_{N-1}, 1) \cdot X_j)} = [1, a_{N-1}, a_{N-2}, \dots] + [1, 5, 2, a_{N+3}, a_{N+4}, \dots].$$

According to the shape of the small graph $\mathcal{S}$ the possible values for $a_{N+3}$ are $a_{N+3} = 4$ and $a_{N+3} = 5$. Similarly, the possible values for $a_{N-1}$ are $a_{N-1} = 4$ and $a_{N-1} = 5$. According to Lemma 4.4 we have

$$D(N,1,\alpha) \geq [1, 4, a_{N-2}, \dots] + [1, 5, 2, 4, a_{N+4}, \dots].$$

Moreover, if $a_{N-1} = 4$ and $a_{N+3} = 4$, then we must have $g(a_1, \dots, a_{N-2}) \cdot X_j = E_3$ and $g(a_1, \dots, a_{N+3}) \cdot X_j = C_6$, according to the shape of $\mathcal{S}$. The only arrow of $\mathcal{S}$ ending in $E_3$ is $C_6 \xrightarrow{1+} E_3$, therefore

$$D(N,1,\alpha) \geq [1, 4, 1, a_{N-3}, \dots] + [1, 5, 2, 4, a_{N+4}, \dots]$$

with $g(a_1,\dots,a_{N-3})\cdot X_j = g(a_1,\dots,a_{N+3})\cdot X_j = C_6$, in other words we have the block

$$C_6 \xrightarrow{1+} E_3 \xrightarrow{4-} C_5 \xrightarrow{2+} C_5 \xrightarrow{5-} C_5 \xrightarrow{2+} C_5 \xrightarrow{4-} C_6.$$

According to the shape of the small graph $\mathcal{S}$ and to Lemma 4.4, in order to minimize $D(N,1,\alpha)$ we must complete the block above repeating both in the past and in the future the block $C_6 \xrightarrow{1+} E_3 \xrightarrow{3-} C_6$. The Lemma follows recalling that $\delta > 0$ is arbitrarily small and observing that we get

$$L(X_j,\alpha) > 7\cdot D(N,1,\alpha) + \delta \geq 7\cdot\left([1,4,\overline{1,3}] + [1,5,2,4,\overline{1,3}]\right) + \delta > \eta_1,$$

which is absurd. $\square$

4.5. **Proof of Theorem 1.2.** The proof splits in two parts.

4.5.1. *Coding by the subshift.* Let $(X_j,\alpha)$ be data with $L(X_j,\alpha) < \eta_1$. According to Proposition 4.3, modulo replacing the path by its image under the involution $\psi$, the arrows that can appear infinitely many times in the path generated by $(X_j,\alpha)$, are the five arrows $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_{C4}$, $\mathcal{F}_{E4}$ of the small graph $\mathcal{S}$. In particular, any path as above necessarily passes infinitely many times from the vertex $C_6$, thus it can be eventually decomposed as $H_0 * H_1 * H_2 \dots$, where any $H_i$ is one of the two elementary loops $H_a$ and $H_b$ at $C_6$ which are defined below and represented in Figure 4.5.1.

$$H_a := C_6 \xrightarrow{1+} E_3 \xrightarrow{4-} C_5 \xrightarrow{2+} C_5 \xrightarrow{4-} C_6$$
$$H_b := C_6 \xrightarrow{1+} E_3 \xrightarrow{3-} C_6.$$

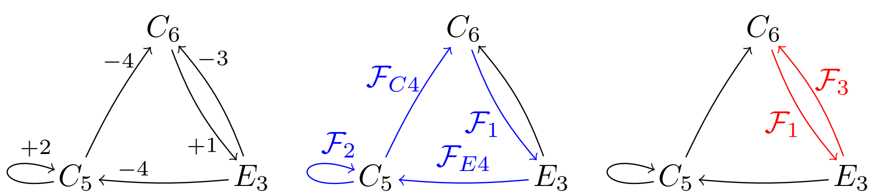


FIGURE 7. The *small graph* $\mathcal{S}$ and the two elementary loops $H_a$ and $H_b$.

One can also see that those paths which pass from $C_5$ just finitely many times eventually coincide with the periodic loop $(C_6,[\overline{1,3}])$, and we already know that $L(C_6,[\overline{1,3}]) = \phi_1$. Therefore we consider without loss of generality the set $\Gamma$ of paths $\gamma$ starting at $C_6$, composed by the five arrows above, containing infinitely many times both the loops $H_a$ and $H_b$, and whose first loop is $H_0 = H_a$. Consider the two finite words $a := 1,4,2,4$ and $b := 1,3$ (we will think of them as a finite block of digits in $\{1,2,3,4\}$). Let $\sigma : \Xi_0 \to \Xi_0$ be the map defined in the introduction, where $\Xi_0$ is the set of $\{a,b\}^{\mathbb{Z}}$ of those sequences $\xi = (\xi_i)_{i\in\mathbb{Z}}$ such that $\xi_0 = a$ and also $\xi_i = a$ for arbitrarily big integers $i > 0$. Define a map $\Pi : \Gamma \to \Xi_0$, where for any $\gamma \in \Gamma$ the sequence $(\xi_i)_{i\in\mathbb{Z}} = \Pi(\gamma)$ is defined setting $\xi_i := a$ for $i \leq 0$ and for all positive integer $i$ setting

$$\xi_i := a \text{ if } H_i = H_a \text{ and}$$
$$\xi_i := b \text{ if } H_i = H_b,$$

where $\gamma = H_1 * H_2 * H_3 * \dots$ is the decomposition of $\gamma$ in loops $H_i \in \{H_a, H_b\}$. Recall that for any sequence $\xi \in \Xi_0$ we write $[\xi]_+ := [1,4,\xi_1,\xi_2,\dots]$ and $[\xi]_- := [1,4,\xi_{-1},\xi_{-2},\dots]$. Consider the words $< a >:= 4,2,4,1$ and $< b >:= 3,1$, then define an operation $< \cdot >$ on the set of finite words $u = \xi_1,\dots,\xi_k$ in the letters $a, b$ setting

$$< \xi_1,\dots,\xi_k >:=< \xi_1 >,\dots,< \xi_k > .$$

Observe that $1,< a >= 1,4,2,4,1 = a,1$ and $1,< b >= 1,3,1 = b,1$, thus for any finite word $u$ we have the identity

$$1,< u >= u,1.$$

Finally, given two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ we write $x_n \sim y_n$ if $x_n - y_n \to 0$ for $n \to +\infty$.

**Lemma 4.15.** *Let $(X_j, \alpha)$ be data generating a path $\gamma \in \Gamma$ and set $\xi = \Pi(\gamma)$. Then at any occurrence of $\mathcal{F}_2$, that is for any $n$ with $g(a_1, \dots, a_{n-1}) \cdot X_j = C_5$ and $a_n = 2$ we have*

$$\frac{D(n,1,\alpha)}{m^2(R \cdot g(a_1, \dots, a_{n-1}, 1) \cdot X_j)} \sim \left[\sigma^{k(n)}(\xi)\right]_+ + \left[\sigma^{k(n)}(\xi)\right]_-,$$

*where the positive integer $k(n)$ corresponds to the number of occurrences of the loop $H_a$ in the segment of the path $\gamma$ corresponding to $g(a_1, \dots, a_n)$.*

*Proof.* Observe that any occurrence in the path $\gamma$ of the arrow $\mathcal{F}_2 = C_5 \xrightarrow{2+} C_5$ corresponds to an occurrence of the block

$$C_6 \xrightarrow{1+} E_3 \xrightarrow{4-} C_5 \xrightarrow{2+} C_5 \xrightarrow{4-} C_6 \xrightarrow{1+} E_3.$$

Therefore, for any integer $n$ as in the statement we have $a_{n-2} = 1$, $a_{n-1} = 4$, $a_n = 2$, $a_{n+1} = 4$ and $a_{n+2} = 1$. Moreover we have $g(a_1, \dots, a_{n-1}, 1) \cdot X_j = C_4$, and since $m^2(R \cdot C_4) = 1$ then we get

$$\frac{D(n,1,\alpha)}{m^2(R \cdot g(a_1, \dots, a_{n-1}, 1) \cdot X_j)} = [1,4,1,a_{n-3}, \dots, a_1] + [1,4,1,a_{n+3}, \dots].$$

Let $k = k(n)$ be the integer in the statement, which corresponds to the number of occurrences of the loop $H_a$ in the segment of the path $\gamma$ corresponding to $g(a_1, \dots, a_n)$. If $\gamma = H_0 * H_1 * H_2 \dots$ is the decomposition of $\gamma$ into loops $H_i \in \{H_a, H_b\}$, then the map $\Pi$ gives

$$[1,4,1,a_{n+3}, \dots] = [1,4,\xi_{k+1},\xi_{k+2} \dots] = \left[\sigma^k(\xi)\right]_+.$$

With the same argument, and recalling that any finite word $u$ in the letters $a, b$ satisfies the identity $1, < u >= u, 1$, we have

$$[1,4,1,a_{n-3}, \dots, a_1] = [1,4,1,<\xi_{k-1}>, \dots, <\xi_0>] = [1,4,\xi_{k-1}, \dots, \xi_0, 1] \sim \left[\sigma^k(\xi)\right]_-.$$

□

4.5.2. *End of the proof of Theorem 1.2.* Let $(X_j, \alpha)$ be data with $L(X_j, \alpha) < \eta_1$ and assume without loss of generality that they generate a path $\gamma \in \Gamma$.

We have $m^2(R \cdot C_6) = 1$ and the only arrow starting at $C_6$ is $\mathcal{F}_1$. According to Formula (3.1) and Lemma 4.4, for the Gauss approximations corresponding to occurrences of $\mathcal{F}_1$ we have the estimation

$$\frac{D(n,1,\alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} < [\overline{3,1}] + 1 + [\overline{3,1}] = \phi_1 = 1,527524 \pm 10^{-6}.$$

We have $m^2(R \cdot C_5) = m^2(R \cdot E_3) = 4$ for all the endpoints of all the remaining four arrows $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_{C4}$ and $\mathcal{F}_{E4}$. Moreover $1 \leq a_n \leq 4$ for any $n$ big enough, therefore for Gauss approximations Formula (3.1) gives the estimation

$$\frac{D(n,a_n,\alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} < \frac{1+4+1}{4} = 1,5.$$

Since $m^2(C_2) = m^2(C_1) = m^2(C_0) = m^2(C_6) = 1$, then Formula (3.1) and Lemma 4.4 give for the Farey approximations corresponding to occurrences of the arrows $\mathcal{F}_{C4}$ and $\mathcal{F}_{E4}$

$$\frac{D(n,1,\alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} < [1,\overline{1,4}] + [3,\overline{1,4}] < 1$$
$$\frac{D(n,2,\alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} < 2[2,\overline{1,4}] < 1$$
$$\frac{D(n,3,\alpha)}{m^2(R \cdot g(a_1, \dots, a_n) \cdot X_j)} < [3,\overline{1,4}] + [1,\overline{1,4}] < 1.$$

Again Formula (3.1) and Lemma 4.4 give for the Farey approximations corresponding to occurrences of the arrow $\mathcal{F}_3$

$$\frac{D(n,1,\alpha)}{m^2(R\cdot g(a_1,\dots,a_n)\cdot X_j)} < [1,1,\overline{3,1}]+[2,1,\overline{3,1}] < 1$$
$$\frac{D(n,2,\alpha)}{m^2(R\cdot g(a_1,\dots,a_n)\cdot X_j)} < [2,1,\overline{3,1}]+[1,1,\overline{3,1}] < 1.$$

So far we proved that for all Gauss approximations and all Farey approximations but those corresponding to the occurrences of the arrow $\mathcal{F}_2$ we have

$$\frac{D(n,1,\alpha)}{m^2(R\cdot g(a_1,\dots,a_n)\cdot X_j)} < \phi_1 \sim 1,527524 \pm 10^{-6}.$$

According to Lemma 4.15 and to Lemma 4.4, for any Farey approximation corresponding to an occurrence of $\mathcal{F}_2$, that is for those $n$ such that $g(a_1,\dots,a_{n-1}) = C_5$ and $a_n = 2$, we have

$$\frac{D(n,1,\alpha)}{m^2(R\cdot g(a_1,\dots,a_{n-1},1)\cdot X_j)} = [1,4,1,a_{n-3},\dots,a_1]+[1,4,1,a_{n+3},\dots]$$
$$\geq [1,4,\overline{1,3}]+[1,4,\overline{1,3}] = 1,654653 \pm 10^{-6}.$$

Theorem 1.2 is proved, since the latter is the bigger of all the values computed above.

## 5. Proof Theorem 1.1

In this paragraph we prove Theorem 1.1 using the function $L^\sigma : \Xi_0 \to \mathbb{R}_+$ in Theorem 1.2. Let $\mathbb{K} := L^\sigma(\Xi_0)$ be the set of values of the function. We believe that Theorem 1.1 can be improved proving that $\mathbb{K}$ is a cantor set. Therefore our proof is presented as an iterative construction of a Cantor set as $\mathbb{K} := \bigcap_{n=1}^\infty \mathbb{K}(n)$ where each $\mathbb{K}(n)$ is a countable union of closed intervals satisfying $\mathbb{K}(n+1) \subset \mathbb{K}(n)$ for any $n \geq 1$. The two families $\mathbb{K}(1)$ and $\mathbb{K}(2)$ are described in §5.2 and §5.3 respectively. Finally in § 5.4 we briefly rephrase results as in the statement of Theorem 1.1.

5.1. **Lexicographic order on finite words in the letters $a, b$.** Let $u$ be a finite word in the letters $a, b$. Let $u^{(k)}$ be the finite word obtained as concatenation $u * \cdots * u$ of $k$ copies of the word $u$. Let $u^{(\infty,+)}$ and $u^{(\infty,-)}$ be respectively the positive infinite word and the negative infinite word obtained concatenating periodically the word $u$. Let $u^\infty$ be the periodic infinite word whose period is $u$, so that for instance we have $u^\infty = u^{(\infty,-)} * u^{(\infty,+)}$. Let $v$ be an other finite word in the letter $a, b$. Let $\overline{v^\infty u v^\infty}$ be the infinite word obtained by concatenation

$$\overline{v^\infty u v^\infty} := u^{(\infty,-)} * v * u * v * v^{(2)} * u * v^{(2)} * \cdots * v^{(k)} * u * v^{(k)} * \dots,$$

that is, the infinite word whose future is the concatenation of blocks $v^{(k)} * u * v^{(k)}$ with increasing values of $k$ and whose past is the half-periodic word $u^{(\infty,-)}$. Actually, since we will consider iterates $\sigma^n(\xi)$ with $n \to +\infty$ of sequences $\xi \in \Xi_0$, the past of $\xi$ does not affect $L^\sigma(\xi)$, and thus in particular one could define $\overline{v^\infty u v^\infty}$ replacing $u^{(\infty,-)}$ by any other half-infinite sequence in the letters $a, b$.

We first give a numerical result, which depends on the explicit values of the finite words $a, b$.

**Lemma 5.1.** *We have the inequalities*

$$2[a, b^{(\infty,+)}] > [b, a^{(\infty,+)}] + [a^{(\infty,+)}]$$
$$2[b, a^{(\infty,+)}] < [a, b^{(\infty,+)}] + [b^{(\infty,+)}].$$

*Proof.* The Lemma just follows comparing the values

$$[b^{(\infty,+)}] = 0,79128784 \pm 10^{-8}$$
$$[b, a^{(\infty,+)}] = 0,79238557 \pm 10^{-8}$$
$$[a, b^{(\infty,+)}] = 0,81660638 \pm 10^{-8}$$
$$[a^{(\infty,+)}] = 0,81661395 \pm 10^{-8}.$$

□

**Lemma 5.2** (Lexicographic Order)**.** *Let $u$ be a finite word in the letters $a, b$. For any positive infinite words $\omega$ and $\omega'$ in the letters $a, b$ we have the strict inequality*

$$[1, 4, u, b, \omega] < [1, 4, u, a, \omega'].$$

*Proof.* Let $g \in \mathrm{PSL}(2, \mathbb{Z})$ be the homography such that $g([\omega'']) = [1, 4, u, \omega'']$ for any positive infinite word $\omega''$. The Lemma corresponds to prove that

$$g([b, \omega]) < g([a, \omega'])$$

for any positive infinite words $\omega$ and $\omega'$. Since $g$ is increasing monotone, the last inequality is equivalent to $[b, \omega] < [a, \omega']$, which follows from Lemma 5.1 observing that

$$[b, \omega] \leq [b, a^{(\infty,+)}] < [a, b^{(\infty,+)}] \leq [a, \omega'].$$

□

In the following we will use frequently the Lemma below, whose simple proof is left to the reader.

**Lemma 5.3.** *Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave and increasing monotone function. If $x_1, x_2, x_3, x_4$ are points in $\mathbb{R}_+$ such that*

$$x_1 = \min_{i=1,2,3,4} x_i$$
$$x_4 = \max_{i=1,2,3,4} x_i$$
$$x_2 + x_3 > x_1 + x_4.$$

*Then we have*

$$g(x_2) + g(x_3) > g(x_1) + g(x_4).$$

Let $u$ be any finite word in the letters $a, b$ and observe that the assumption of Lemma 5.3 are satisfied by the homography $g$ such that $g([\omega]) = [1, 4, u, \omega]$ for any positive infinite words $\omega$, indeed $g$ is increasing as element of $\mathrm{PSL}(2, \mathbb{Z})$, moreover its pole $g^{-1}(\infty)$ is negative, since all coefficients of $g$ are positive, thus $g$ is positive and concave on $\mathbb{R}_+$.

**Lemma 5.4.** *Consider a pair of integers $i, j \in \mathbb{N}$ such that either $i = j$ or $|i - j| = 1$. Then for any pair of integers $i', j' \in \mathbb{N}$ with $|i' + j'| = |i + j|$ and $|i' - j'| > |i - j|$ and for any positive infinite words $\omega$ and $\omega'$ in the letters $a, b$ we have the strict inequality*

$$[1, 4, a^{(i)}, b, \omega] + [1, 4, a^{(j)}, b, \omega'] > [1, 4, a^{(i')}, b, \omega] + [1, 4, a^{(j')}, b, \omega']$$

*Proof.* Assume without loss of generality that $j' < j \leq i < i'$ and consider the positive integer $l := i' - i \geq 1$. Observe that we have also $j - j' = l$ since $i + j = i' + j'$. Consider the homography $g$ such that $g([\omega'']) = [1, 4, a^{(j')}, \omega'']$ for any positive infinite word $\omega''$. Observe that $g$ is increasing monotone, since it belongs to $\mathrm{PSL}(2, \mathbb{Z})$, and that $g^{-1}(\infty) < 0$, since $g$ has positive coefficients. The Lemma follows from

$$g([a^{(l)}, b, \omega]) + g([a^{(l)}, b, \omega']) > g([a^{(2l)}, b, \omega]) + g([b, \omega']) \text{ if } i = j$$
$$g([a^{(l+1)}, b, w]) + g([a^{(l)}, b, w']) > g([a^{(2l+1)}, b, w]) + g([b, w']) \text{ if } i = j + 1.$$

According to Lemma 5.2 and recalling that $g$ is increasing monotone, it is enough to prove that for any $l \geq 1$ we have

$$g([a^{(l)}, b^{(\infty,+)}]) + g([a^{(l)}, b^{(\infty,+)}]) > g([a^{(2l)}, b, a^{(\infty,+)}]) + g([b, a^{(\infty,+)}])$$
$$g([a^{(l+1)}, b^{(\infty,+)}]) + g([a^{(l)}, b^{(\infty,+)}]) > g([a^{(2l+1)}, b, a^{(\infty,+)}]) + g([b, a^{(\infty,+)}])$$

and since $g$ is concave and strictly increasing, according to Lemma 5.3 it is enough to prove that for any $l \geq 1$ we have

$$[a^{(l)}, b^{(\infty,+)}] + [a^{(l)}, b^{(\infty,+)}] > [a^{(2l)}, b, a^{(\infty,+)}] + [b, a^{(\infty,+)}]$$
$$[a^{(l+1)}, b^{(\infty,+)}] + [a^{(l)}, b^{(\infty,+)}] > [a^{(2l+1)}, b, a^{(\infty,+)}] + [b, a^{(\infty,+)}].$$

Since $l \geq 1$, according to Lemma 5.2 it is enough to prove

$$2[a^{(1)}, b^{(\infty,+)}] > [a^{(\infty,+)}] + [b, a^{(\infty,+)}]$$
$$[a^{(2)}, b^{(\infty,+)}] + [a^{(1)}, b^{(\infty,+)}] > [a^{(\infty,+)}] + [b, a^{(\infty,+)}].$$

The first inequality corresponds to the statement of Lemma 5.1. The second inequality follows from the first observing that $[a^{(2)}, b^{(\infty,+)}] > [a^{(1)}, b^{(\infty,+)}]$, according to Lemma 5.2. □

5.2. **The first generation of $\mathbb{K}$.** Consider the function $\kappa : \Xi_0 \to \mathbb{N}^*$, where for any $\xi \in \Xi_0$ the integer $\kappa(\xi)$ is the maximum $k \geq 1$ such that for any $N \in \mathbb{N}$ there exists $n > N$ with $\xi_j = a$ for all $j$ with $n \leq j \leq n + k - 1$.

**Lemma 5.5.** *Consider $\xi \in \Xi_0$ such that $\kappa(\xi) = k$. Then there exist positive infinite words $\omega = \omega(\xi)$ and $\omega' = \omega'(\xi)$ in the letters $a, b$ such that*

$$L^\sigma(\xi) = 7 \cdot \left([1, 4, a^{(i)}, b, \omega'] + [1, 4, a^{(i)}, b, \omega]\right) \text{ if } k = 2i + 1$$
$$L^\sigma(\xi) = 7 \cdot \left([1, 4, a^{(i+1)}, b, \omega'] + [1, 4, a^{(i)}, b, \omega]\right) \text{ if } k = 2i + 2.$$

*Proof.* There exist two positive infinite words $\omega$ and $\omega'$ such that

$$L^\sigma(\xi) = 7 \cdot \left([1, 4, \omega'] + [1, 4, \omega]\right).$$

The Lemma follows from Lemma 5.2 and Lemma 5.4. □

We observe that the two terms containing respectively in $\omega$ and $\omega'$ in Lemma 5.5 does not necessarily correspond to the *past* or to the *future*. The same ambiguity appears further in Lemma 5.8.

**Lemma 5.6.** *The first generation $\mathbb{K}(1)$ is the set of closed intervals $I_k$ with $k \in \mathbb{N}^*$ defined by*

$$I_k := \left[L^\sigma(\overline{b^\infty a^{(k)} b^\infty}), L^\sigma((ba^{(k)})^\infty)\right].$$

*Moreover, for any $k \in \mathbb{N}$ and any $\xi \in \Xi_0$ we have $L^\sigma(\xi) \in I_k$ if and only if $\kappa(\xi) = k$. Finally, the last two conditions are equivalent to the existence of positive infinite words $\omega$ and $\omega'$, depending on $\xi$ and $k$, such that*

$$L^\sigma(\xi) = 7 \cdot \left([1, 4, a^{(i)}, b, \omega'] + [1, 4, a^{(i)}, b, \omega]\right) \text{ if } k = 2i + 1$$
$$L^\sigma(\xi) = 7 \cdot \left([1, 4, a^{(i)}, b, \omega'] + [1, 4, a^{(i+1)}, b, \omega]\right) \text{ if } k = 2i + 2.$$

As it is described in §5.4 below, the first generation of gaps $(G_k)_{k \geq 1}$ is given by the connected components of the complement of $\mathbb{K}(1)$.

*Proof.* We prove that for any integer $k \geq 0$ we have a gap in $\mathbb{K}$ given by the open interval

$$\left(L^\sigma((ba^{(k)})^\infty), L^\sigma(\overline{b^\infty a^{(k+1)} b^\infty})\right).$$

We assume that $k$ is odd, the proof for even $k$ being the same. Set $k = 2i + 1$ with $i \in \mathbb{N}$. Since both $\kappa\left(L^\sigma(b(a^{(k)})^\infty)\right) = k$ and $\kappa\left(\overline{b^\infty a^{(k+1)} b^\infty}\right) = k$, Lemma 5.5 implies

$$L^\sigma\left((ba^{(k)})^\infty\right) = 7 \cdot \left([1, 4, a^{(i)}, (ba^{(k)})^{(\infty,+)}] + [1, 4, a^{(i)}, (ba^{(k)})^{(\infty,+)}]\right)$$
$$L^\sigma(\overline{b^\infty a^{(k+1)} b^\infty}) = 7 \cdot \left([1, 4, a^{(i)}, b^{(\infty,+)}] + [1, 4, a^{(i+1)}, b^{(\infty,+)}]\right).$$

Comparing the two quantities above with the expression in Lemma 5.5 one gets

$$L^\sigma\left((ba^{(k)})^\infty\right) = \max\{L^\sigma(\xi) \; ; \; \xi \in \Xi_0, \kappa(\xi) = k\}$$
$$L^\sigma(\overline{b^\infty a^{(k+1)} b^\infty}) = \min\{L^\sigma(\xi) \; ; \; \xi \in \Xi_0, \kappa(\xi) = k + 1\}.$$

The Lemma follows proving that we have the strict inequality

$$L^\sigma\left((ba^{(k)})^\infty\right) < L^\sigma(\overline{b^\infty a^{(k+1)} b^\infty}).$$

According to the formulae for $L^\sigma\left((ba^{(k)})^\infty\right)$ and $L^\sigma(\overline{b^\infty a^{(k+1)} b^\infty})$ obtained above, and observing that Lemma 5.2 implies

$$L^\sigma\left((ba^{(k)})^\infty\right) = 7 \cdot \left([1, 4, a^{(i)}, (ba^{(k)})^{(\infty,+)}] + [1, 4, a^{(i)}, (ba^{(k)})^{(\infty,+)}]\right) \leq 14 \cdot [1, 4, a^{(i)}, b, a^{(\infty,+)}]$$

it is enough to prove that

$$2 \cdot [1,4,a^{(i)},b,a^{(\infty,+)}] < [1,4,a^{(i)},b^{(\infty,+)}] + [1,4,a^{(i+1)},b^{(\infty,+)}].$$

In order to prove the last inequality, let $g$ be the homography such that $g([\omega]) = [1,4,a^{(i)},\omega]$ for any positive infinite word $\omega$. The inequality is equivalent to

$$g\big([b,a^{(\infty,+)}]\big) - g\big([b^{(\infty,+)}]\big) < g\big([a,b^{(\infty,+)}]\big) - g\big([b,a^{(\infty,+)}]\big)$$

and the latter follows from Lemma 5.3. □

5.3. **The second generation of $\mathbb{K}$.** Let $\Xi_{0,k}$ be the set of $\xi \in \Xi_0$ with $\kappa(\xi) = k$. Define a function $\nu : \Xi_{0,k} \to \mathbb{N}$, where for any $\xi \in \Xi_{0,k}$ the integer $\nu(\xi)$ is the minimum positive integer $n$ such that in $\xi$ we have infinitely often one of the two finite words

$$\text{either } a, b^{(n)}, a^{(k)} \text{ or } a^{(k)}, b^{(n)}, a.$$

**Lemma 5.7.** *For any $i \in \mathbb{N}$, any positive integer $n$ and any pair of positive infinite words $\omega$ and $\omega'$ in he letters $a, b$ we have*

$$[1,4,a^{(i+1)},b^{(n)},\omega] + [1,4,a^{(i)},b^{(n)},a,\omega'] > [1,4,a^{(i)},b^{(n)},\omega] + [1,4,a^{(i+1)},b^{(n)},a,\omega'].$$

*Proof.* Let $h$ and $g$ be the homographies such that we have respectively $h([\omega'']) = [a,\omega'']$ and $g([\omega'']) = [1,4,a^{(i)},\omega'']$ for any positive infinite word $\omega''$. Since $0 < h'(t) < 1$ for any $h \in \mathrm{PSL}(2,\mathbb{Z})$ and any $t > 0$ then we have

$$[b^{(n)},a,\omega'] - [b^{(n)},\omega] > h\big([b^{(n)},a,\omega']\big) - h\big([b^{(n)},\omega]\big),$$

that is

$$[a,b^{(n)},\omega] + [b^{(n)},a,\omega'] > [b^{(n)},\omega] + [a,b^{(n)},a,\omega'].$$

The statement follows applying Lemma 5.3 to the function $g$. □

The same argument on the proof of Lemma 5.5 and the estimate in Lemma 5.7 give the Lemma below.

**Lemma 5.8.** *Consider $\xi \in \Xi_{0,k}$ such that $\nu(\xi) = n$. Then there exist positive infinite words $\omega = \omega(\xi)$ and $\omega' = \omega'(\xi)$ in the letters $a, b$ such that*

$$L^\sigma(\xi) = 7 \cdot \big([1,4,a^{(i)},b^{(n)},\omega'] + [1,4,a^{(i)},b^{(n)},a,\omega]\big) \text{ if } k = 2i+1$$
$$L^\sigma(\xi) = 7 \cdot \big([1,4,a^{(i+1)},b^{(n)},\omega'] + [1,4,a^{(i)},b^{(n)},a,\omega]\big) \text{ if } k = 2i+2.$$

**Lemma 5.9.** *For any $I_k \in \mathbb{K}(1)$ the generation $\mathbb{K}(2|I_k)$ is the family of closed intervals $J_n$ for $n \in \mathbb{N}^*$ defined by*

$$J_n = \big[L^\sigma(\overline{b^\infty a^{(k)} b^{(n)} a b^\infty}), L^\sigma((a^{(k)} b^{(n)})^\infty)\big].$$

*Moreover, for any $n \in \mathbb{N}^*$ and any $\xi \in \Xi_{0,k}$ we have $L^\sigma(\xi) \in J_n$ if and only if $\nu(\xi) = n$. Finally, the last two conditions are equivalent to the existence of positive infinite words $\omega$ and $\omega'$, depending on $\xi$ such that*

$$L^\sigma(\xi) = 7 \cdot \big([1,4,a^{(i)},b^{(n)},\omega'] + [1,4,a^{(i)},b^{(n)},a,\omega]\big) \text{ if } k = 2i+1$$
$$L^\sigma(\xi) = 7 \cdot \big([1,4,a^{(i+1)},b^{(n)},\omega'] + [1,4,a^{(i)},b^{(n)},a,\omega]\big) \text{ if } k = 2i+2.$$

As it is described in §5.4 below, for any $I_k \in \mathbb{K}(1)$ the second generation of gaps $(G_{k,n})_{n\geq 1}$ in the interval $I_k$ is given by the connected components of the complement of $\mathbb{K}(2|I_k)$.

*Proof.* We prove that for any $n \geq 0$ we have a gap in $I_k$ corresponding to the open interval

$$\big(L^\sigma((a^{(k)} b^{(n)})^\infty), L^\sigma(\overline{b^\infty a^{(k)} b^{(n+1)} a b^\infty})\big).$$

We assume that $k$ is even, the proof for even $k$ being the same (the assumption is complementary to the one in the proof of Lemma 5.6, where only the case of odd $k$ is considered explicitly). Set $k = 2i+2$

with $i \in \mathbb{N}$. Observe that both $L^\sigma((a^{(k)}b^{(n)})^\infty)$ and $L^\sigma(\overline{b^\infty a^{(k)}b^{(n+1)}ab^\infty})$ belong to $I_k$, therefore Lemma 5.6 implies

$$L^\sigma\big((a^{(k)}b^{(n+1)})^\infty\big) = 7 \cdot \big([1,4,a^{(i+1)},(b^{(n+1)}a^{(k)})^{(\infty,+)}] + [1,4,a^{(i)},(b^{(n+1)}a^{(k)})^{(\infty,+)}]\big)$$
$$L^\sigma(\overline{b^\infty a^{(k)}b^{(n)}ab^\infty}) = 7 \cdot \big([1,4,a^{(i+1)},b^{(\infty,+)}] + [1,4,a^{(i)},b^{(n)},a,b^{(\infty,+)}]\big),$$

where in the second equality we also need the estimate in Lemma 5.7. Lemma 5.8 implies

$$L^\sigma\big((a^{(k)}b^{(n+1)})^\infty\big) = \max\{L^\sigma(\xi)\ ;\ \xi \in \Xi_{0,k}, \nu(\xi) = n+1\}$$
$$L^\sigma(\overline{b^\infty a^{(k)}b^{(n)}ab^\infty}) = \min\{L^\sigma(\xi)\ ;\ \xi \in \Xi_{0,k}, \nu(\xi) = n\}.$$

The Lemma follows proving that we have the strict inequality

$$L^\sigma\big((a^{(k)}b^{(n+1)})^\infty\big) < L^\sigma(\overline{b^\infty a^{(k)}b^{(n)}ab^\infty}).$$

According to the expressions of these two quantities obtained above and observing that Lemma 5.2 implies

$$L^\sigma\big((a^{(k)}b^{(n+1)})^\infty\big) < 7 \cdot \big([1,4,a^{(i+1)},b^{(n+1)},a^{(\infty,+)}] + [1,4,a^{(i)},b^{(n+1)},a^{(\infty,+)}]\big)$$
$$L^\sigma(\overline{b^\infty a^{(k)}b^{(n)}ab^\infty}) = 7 \cdot \big([1,4,a^{(i+1)},b^{(\infty,+)}] + [1,4,a^{(i)},b^{(n)},a,b^{(\infty,+)}]\big)$$

it is enough to prove

$$g\big([a,b^{(n+1)},a^{(\infty,+)}]\big) + g\big([b^{(n+1)},a^{(\infty,+)}]\big) < g\big([a,b^{(\infty,+)}]\big) + g\big([b^{(n)},a,b^{(\infty,+)}]\big),$$

where $g$ is the homography such that $g([\omega]) = [1,4,a^{(i)},\omega]$ for any positive infinite word $\omega$. □

5.4. **End of the proof of Theorem 1.1.** Theorem 1.1 simply follows rephrasing the statements of Lemma 5.6 and of Lemma 5.9. The first generation $(G_k)_{k\geq 1}$ of gaps in $\mathcal{L}(S)$ is given by

$$G_0 := (\phi_1,\phi_2) \text{ and}$$
$$G_k := \big(L^\sigma((ba^{(k)})^\infty), L^\sigma(\overline{b^\infty a^{(k+1)}b^\infty})\big) \text{ for } k \geq 1.$$

For any interval $I_k = \big[L^\sigma(\overline{b^\infty a^{(k)}b^\infty}), L^\sigma((ba^{(k)})^\infty)\big]$ of $\mathbb{K}(1)$ with $k \geq 1$ the second generation of gaps $(G_{k,n})_{n\geq 1}$ in the interval $I_k$ is the family of intervals defined by

$$G_{k,n} := \big(L^\sigma((a^{(k)}b^{(n+1)})^\infty), L^\sigma(\overline{b^\infty a^{(k)}b^{(n)}ab^\infty})\big) \text{ for } n \geq 1.$$

According to Lemma 5.9, and assuming that $k = 2i$, we have

$$G_k^{(+)} = L^\sigma\big(b^\infty a^{(k+1)}b^\infty\big) = 14 \cdot [1,4,a^{(i)},b^\infty]$$
$$G_{k+1}^{(-)} = L^\sigma\big((ba^{(k+1)})^\infty\big) = 14 \cdot [1,4,a^{(i)},(ba^{(k+1)})^\infty],$$

therefore $G_k^{(+)} < G_{k+1}^{(-)}$ is equivalent to $[b^\infty] < [(ba^{(k+1)})^\infty]$, which is true according to Lemma 5.2. From the same Lemma it is obvious that for $k = 2i$ and $i \to +\infty$ we have

$$G_k^{(+)} = 14 \cdot [1,4,a^{(i)},b^\infty] \to 14 \cdot [1,4,a^\infty] = \phi_\infty.$$

The same estimate holds for $k = 2i+1$. A similar argument proves the analogous statement for the second generation of holes $(G_{k,n})_{n\geq 1}$, where for any $k \geq 1$ and $n \geq 1$ the formula for the endpoints $G_{k,n}(-)$ and $G_{k,n}^{(+)}$ is given by Lemma 5.9. Theorem 1.1 is proved.

**Acknowledgements.** We are grateful to the anonymous referee for his/her careful reading and useful comments. C. Ulcigrai is currently supported by the ERC Grant ChaParDyn. The authors would like to thank the hospitality given by the Max Planck Institute during the program *Dynamics and Numbers* and by CIRM in Luminy, Marseille, during the conference *Dynamics and Geometry in the Teichmueller Space*, where parts of this project were completed. Research visits which made this collaboration possible were also supported by the EPSRC Grant EP/I019030/1, and by the ERC Grant ChaParDyn. The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n. 335989.

## References

[ArMarUl] M. Artigiani, L. Marchese, C. Ulcigrai: *The Lagrange spectrum of a Veech surface has Hall ray.* Arxiv:1409.7023.

[BoDe] M. Boshernitzan, V. Delecroix: *A packing problem, quantitative recurrence in [0,1] and the bottom of the Lagrange spectrum for interval exchange transformations.* In preparation.

[CeMatMo] A. Cerqueira, C. Matheus and C. G. Moreira *Continuity of Hausdorff dimension across dynamical Lagrange and Markow spectra.* Arxiv:1605:09037.

[CuFl] T. W. Cusick, M. E. Flahive: *The Markoff and Lagrange Spectra.* Mathematicas Surveys and Monographs, 30, 1989.

[FoMat] G. Forni, C. Matheus: *Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations flows on surfaces and billiards.* Arxiv:1311.2758.

[Ha] M. Hall: *On the sum and products of continued fractions.* Annals of Math, 48, (1947), 966-993.

[HuLe] P. Hubert, S. Lelièvre: *Prime arithmetic Teichmüller disc in* $\mathcal{H}(2)$. Israel J. Math. 151 (2006), 281-321.

[HuMarUl] P. Hubert, L. Marchese, C. Ulcigrai: *Lagrange spectra in Teichmüller dynamics via renormalization*, Geom. Funct. Anal. 25 (2015), no.1, 180-255.

[IbMo] S. Ibarra and C. G. Moreira: *On the Lagrange and Markov Dynamical Spectra for Geodesic Flows in Surfaces with Negative Curvature.* Preprint arXiv:1505.05178.

[Mc] C. T. McMullen: *Teichmüller curves in genus two: discriminant and spin*, Math. Ann. 333 (2005), 87-130.

[Mo] C. G. Moreira: *Geometric properties of Markov and Lagrange spectra.* Preprint IMPA.

[ParPau] J. Parkkonen; F. Paulin: *Prescribing the behaviour of geodesics in negative curvature.* Geom. Topol. 14 (2010), no. 1, 277-392.

[Se] C. Series: *The geometry of Markoff numbers.* Math. Intelligencer 7 (1985), 20-29.

[SmWe] J. Smillie, B. Weiss: *Finiteness results for flat surfaces: a survey and problem list.* Partially Hyperbolic Dynamics, Laminations, and Teichmüller Flow. Proceedings of a Conference. Fields Institute, Toronto (2006)

[Ve] W. Veech: *Geometric realizations of hyperelliptic curves.* Algorithms, Fractals, and Dynamics. Okayama Kyoto, 1992, 217?226. Plenum, New York (1995)

[Zo] A. Zorich: *Flat surfaces.* Frontiers in number theory, physics, and geometry. I, 437-583, Springer, Berlin, 2006.

I2M, Centre de Mathématiques et Informatique (CMI), Université Aix-Marseille, 39 rue Joliot Curie, 13453 Marseille Cedex 13, France.

*E-mail address*: pascal.hubert@univ-amu.fr

Laboratoire de mathématique d'Orsay, UMR 8628 CNRS Université Paris-Sud, Bâtiment 425, 91405 Orsay cedex, France

*E-mail address*: samuel.lelievre@math.u-psud.fr

Université Paris 13, Sorbonne Paris Cité, LAGA, UMR 7539, 99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse, France.

*E-mail address*: marchese@math.univ-paris13.fr

School of Mathematics, University of Bristol, University Walk, Clifton, Bristol, BS8 1TW, United Kingdom.

*E-mail address*: corinna.ulcigrai@bristol.ac.uk